

Classification of Quantum Physical Systems

Part VI of *The Classification of Mathematical Representations of Physical Systems*

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Abstract

Quantum physical systems admit a complete structural classification only in restricted regimes, and this paper records where those regimes begin and end. Finite-dimensional observable algebras are classified by one combinatorial invariant, the dimension vector of the block decomposition. Amenable operator-algebraic models are classified by Murray-von Neumann type together with Connes invariants. Completely positive maps admit a canonical normal form through minimal Stinespring dilations, unique up to unitary equivalence, and the Kraus rank of that normal form is not an isomorphism invariant of channels. Beyond the amenable class no finite answer is possible: uncountably many pairwise non-isomorphic separable II_1 factors exist, so no finite list of factor types can be exhaustive. On the categorical side, the Doplicher-Roberts reconstruction theorem and Deligne's theorem classify superselection and symmetric tensor categories under explicit hypotheses, and the Etingof-Gelaki examples show that any equivalence forgetting the symmetric structure fails to recover its group. Quantum monotone metrics form a parametric family rather than a unique choice. The paper separates classified islands from proved impossibilities and open problems, and reports executable models of the finite-dimensional taxonomy as computational evidence only.

1 Introduction

Three kinds of result organize the classification of quantum physical systems: complete classification theorems, proved non-classification theorems, and open problems. The finite-dimensional theory is complete and combinatorial. The amenable operator-algebraic theory is complete and analytic. Outside the amenable class, McDuff's theorem of 1969 rules out any finite list of factor types, and this negative result is content of the theory, not a temporary difficulty. The categorical theory is complete under explicit hypotheses and open without them.

The classification question follows the program of this series. Fix a category of mathematical representations of physical systems and an equivalence relation on its objects; classify the objects up to that equivalence. For quantum systems the representations are operator-algebraic and categorical: Hilbert spaces, C^* -algebras and von Neumann algebras of observables, completely positive maps between them, and tensor categories of superselection sectors. The equivalence relations that appear in established results are exact: $*$ -isomorphism of algebras, unitary equivalence of dilations, monoidal equivalence of categories, each with the structure it preserves made explicit.

Three structural families carry the quantum material: Hilbert spaces and operator algebras, probability and measure structure on the quantum side, and tensor categories with compositional structure. The classifying invariants in force are correspondingly three kinds: a discrete invariant (the dimension vector), a pair of analytic invariants (Murray-von Neumann type and Connes invariants), and a parametric invariant (an operator monotone function classifying a monotone metric). The families are candidate organizing families, and no independence or completeness claim about them is used here.

Two scope restrictions hold throughout. The long-term classification target of the program is a research objective and is never assumed as a hypothesis; every statement proved here is proved from its own hypotheses. The paper classifies mathematical representations of physical systems.

2 Mathematical framework

Definition 2.1 (Quantum system record). A quantum system record is a tuple $P = (H, A, D, S, C)$ where H is a complex Hilbert space, A is a C^* -algebra of observables represented on H , D is the dynamical data (a flow, a semigroup of endomorphisms, or a generator), S is the symmetry and constraint data (a group or groupoid action with its fixed-point structure), and C is the composition data (tensor product or another monoidal operation on records of the same kind). The five components need not be independent.

A morphism of records commutes with the data it claims to preserve, and every equivalence statement below names the preserved layers on its face. The classification results below concern projections of the record: the observable algebra, the channel data, the sector category, and the state-space metric. Theorem 3.3 states what that projection costs for the full tuple.

Definition 2.2 (Classification problem). Fix a class $\mathcal{Q}$ of quantum system records and an equivalence relation $\approx$ on $\mathcal{Q}$ declared with its preserved layers as in theorem 2.6. The classification problem for $(\mathcal{Q}, \approx)$ is to construct an invariant I on $\mathcal{Q}$ with $I(P) = I(Q)$ exactly when $P \approx Q$. The equivalences in force below are $*$ -isomorphism for observable algebras, unitary equivalence for minimal dilations, symmetric tensor equivalence for superselection categories, and equality of the operator monotone parameter for monotone metrics.

Definition 2.3 (Observable algebras). A finite-dimensional quantum system is a record with $\dim H = n < \infty$ and $A \subseteq B(H)$ a C^* -subalgebra of the algebra of bounded operators on H . As an abstract C^* -algebra A is finite-dimensional and unital, with a unit that need not be the identity of $B(H)$; theorem 3.1 applies to A as an abstract algebra. An operator-algebraic system is a record whose observable algebra A is a C^* -algebra or a von Neumann algebra of arbitrary dimension. A factor is a von Neumann algebra with trivial center.

Definition 2.4 (States and channels). A state on a C^* -algebra A is a positive linear functional ρ of norm one. A channel between matrix algebras $\Phi : M_n(\mathbb{C}) \rightarrow M_m(\mathbb{C})$ is a linear map that is completely positive and trace preserving. A Stinespring dilation of a channel Φ is a pair (K, V) with K a finite-dimensional Hilbert space (the environment) and $V : \mathbb{C}^n \rightarrow \mathbb{C}^m \otimes K$ an isometry such that

$$\Phi(X) = \text{Tr}_K(VXV^*) \quad \text{for all } X \in M_n(\mathbb{C}).$$

The dilation is minimal when no proper subspace of K carries a dilation of Φ .

Definition 2.5 (Dimension vector). Let A be a finite-dimensional C^* -algebra. The dimension vector of A is the multiset of matrix block sizes appearing in the decomposition of theorem 3.1, written $\mathbf{d}(A) = \{n_1, \dots, n_k\}$ with each $n_i \geq 1$ and repetition allowed.

Definition 2.6 (Preserved structure of an equivalence). An equivalence used to identify two quantum system records must state which layers it preserves. The layers named in this paper are the $*$ -structure of observable algebras, the monoidal structure of tensor products, the symmetric structure of a tensor category, and the complete positivity of channels. An identification that does not name its preserved layers is inadmissible.

The counterexample of section 7 shows that an equivalence of tensor categories which forgets the symmetric layer cannot recover the group it classifies, so a classification built on such an equivalence is false.

Example 2.7 (Qubit composite systems). A single qubit record has $H = \mathbb{C}^2$ and observables $M_2(\mathbb{C})$. An N -qubit record has $H = (\mathbb{C}^2)^{\otimes N}$ and observables $M_{2^N}(\mathbb{C})$ or a subalgebra of it, with composition data the tensor product of theorem 3.11. The dimension vector of the full observable algebra is $\{2^N\}$, and subalgebras correspond to blocks of smaller size in the sense of theorem 3.1. Symmetry actions such as the Pauli group permute and mix the blocks, and that action is data the dimension vector does not carry.

Definition 2.8 (Invariants). A classifying invariant for a class of records is a map I such that $I(P) = I(Q)$ if and only if P and Q are equivalent under the relation in force. The classifying invariants of this paper are the dimension vector (theorem 2.5), the Murray-von Neumann type and the Connes invariants of a factor, and the operator monotone function classifying a monotone metric on quantum states. The Kraus rank of a minimal dilation is a normal-form dimension and not a classifying invariant for channels (theorem 6.5).

Classification invariants separate at three levels of resolution. The dimension vector completely separates finite-dimensional observable algebras up to $*$ -isomorphism (theorem 3.1). Operator monotone functions parameterize the monotone metrics of quantum states without singling out a unique metric (theorem 8.3). Minimal Stinespring dilations give canonical normal forms for completely positive maps while failing to separate unitarily inequivalent channels (theorem 6.8).

Table 1 records the families in force, their defining invariants, and what the established theorems say about each.

The classes covered below are fixed here: finite-dimensional quantum systems with observable C^* -algebras, operator-algebraic systems with factor observable algebras, channels between matrix algebras, superselection categories of local nets under the DHR selection criterion, and density-matrix state spaces with monotone metrics. Gauge-theoretic and gravitational systems lie outside this scope.

3 Finite-dimensional quantum systems

Finite-dimensional quantum systems are the reference case for the whole program: the classification is complete, the invariant is discrete, and the proofs are self-contained. The observable algebra of such a system is a finite-dimensional C^* -algebra, and the structure theorem below reduces every such algebra to matrix blocks.

Theorem 3.1 (Structure of finite-dimensional C^* -algebras). *Let A be a finite-dimensional complex C^* -algebra. Then there are integers $n_1, \dots, n_k \geq 1$ and a $*$ -isomorphism*

$$A \cong M_{n_1}(\mathbb{C}) \oplus \dots \oplus M_{n_k}(\mathbb{C}).$$

The multiset $\{n_1, \dots, n_k\}$ determines A up to $$ -isomorphism, and two finite-dimensional C^* -algebras are $*$ -isomorphic if and only if their dimension vectors agree as multisets.*

Table 1: Structural families carrying quantum systems, with the classifying data each family actually admits in the established literature.

Family	Object	Classifying invariant	Status
II, Hilbert spaces and operator algebras	Finite-dimensional C^* -algebras	Dimension vector	Complete classification
II, Hilbert spaces and operator algebras	Amenable factors	Murray-von Neumann type and Connes invariants	Complete classification
II, Hilbert spaces and operator algebras	General II_1 factors	No parametric invariant can suffice	Proved non-classifiable
V, probability and measure structure	Monotone metrics on density matrices	Operator monotone function	Parametric classification
VIII, tensor categories and composition	Symmetric tensor categories under the hypotheses of theorem 7.5	Underlying supergroup	Complete under stated hypotheses
VIII, tensor categories and composition	Fusion categories	Weak Hopf presentation only	Open

Proof. The Jacobson radical $J(A)$ of the finite-dimensional algebra A is nilpotent: $J(A)^k = \{0\}$ for some k , so every element of $J(A)$ is nilpotent. For $x \in J(A)$ the element x^*x lies in $J(A)$ and is self-adjoint and nilpotent, hence has spectrum $\{0\}$, and the C^* -identity $\|x\|^2 = \|x^*x\| = r(x^*x)$ gives $x = 0$. Thus $J(A) = \{0\}$ and A is semisimple. By the Artin-Wedderburn theorem over the algebraically closed field $\mathbb{C}$ the algebra decomposes as $A \cong \bigoplus_{i=1}^k M_{n_i}(\mathbb{C})$ with each summand a simple two-sided ideal. Each simple ideal is stable under the involution: if an ideal I satisfied $I \cap I^* = \{0\}$, then every $y \in I$ would satisfy $y^*y \in I \cap I^* = \{0\}$, forcing $y = 0$. Because every conjugate-linear anti-automorphism of $M_{n_i}(\mathbb{C})$ that makes it a C^* -algebra is conjugate through an inner automorphism to the standard adjoint, the algebraic direct sum lifts to a $*$ -isomorphism $A \cong \bigoplus_{i=1}^k M_{n_i}(\mathbb{C})$ with the adjoint taken blockwise. Every finite-dimensional C^* -algebra is unital: the unit is the sum $\sum_i I_{n_i}$ of the block identity matrices, so no unitization step is needed.

For the classification statement, a $*$ -isomorphism carries minimal central projections to minimal central projections and block sizes to block sizes, so the dimension vector is an isomorphism invariant. Conversely, equality of dimension vectors gives a blockwise unitary identification, and this identification intertwines the adjoints blockwise. The dimension vector is therefore a classifying invariant in the sense of theorem 2.8 for the equivalence relation of $*$ -isomorphism. $\square$

Corollary 3.2 (Classification by dimension vector). *Up to $*$ -isomorphism the finite-dimensional complex C^* -algebras are in bijection with finite multisets of positive integers, and every such algebra is unital with $\dim A = \sum_i n_i^2$ the sum of squares of its block sizes.*

Remark 3.3 (What the dimension vector does not classify). Theorem 3.1 classifies the observable algebra only. A designated state, the dynamics implemented by one-parameter groups of automorphisms, symmetry actions, and the composition data of a quantum system record are not determined by the dimension vector. Two system records with isomorphic observable algebras may carry different dynamics and different composition data. The classification of representations is the first layer of the program, not the last.

Example 3.4 (Qubit observables). For $H = \mathbb{C}^2$ the full observable algebra is $M_2(\mathbb{C})$ with dimension vector $\{2\}$. The commutative algebra of diagonal observables is $\mathbb{C} \oplus \mathbb{C}$ with dimension vector $\{1, 1\}$.

The two algebras have equal vector-space dimension 4 and are not $*$ -isomorphic, so vector-space dimension is not a classifying invariant while the dimension vector is. Up to $*$ -isomorphism there are exactly two C^* -algebras of complex dimension 4: $M_2(\mathbb{C})$ and $\mathbb{C}^4$, since $4 = 2^2 = 1^2 + 1^2 + 1^2 + 1^2$ are the two partitions of 4 into perfect squares.

Example 3.5 (Two presentations of one algebra). Let $A \subseteq M_3(\mathbb{C})$ consist of the block matrices $\text{diag}(M, \lambda)$ with $M \in M_2(\mathbb{C})$ and $\lambda \in \mathbb{C}$, and let $B = UAU^*$ for a unitary $U \in M_3(\mathbb{C})$. The algebras are $*$ -isomorphic by construction and have the same dimension vector $\{1, 2\}$, while their matrix representations in $\mathbb{C}^3$ differ. The algebra $M_2(\mathbb{C}) \oplus \mathbb{C}$ carries the same dimension vector, and a unitary carrying the block decomposition of A to the standard one exhibits the $*$ -isomorphism of theorem 3.1 explicitly. The invariant sees the blocks and the presentation hides them.

Corollary 3.6 (Counting small algebras). *For each $d \geq 1$ the $*$ -isomorphism classes of finite-dimensional C^* -algebras of complex dimension d are in bijection with the partitions of d into perfect squares. In particular the number of such algebras grows as the number of such partitions.*

We write a multiset such as eight 1s and one 2 as $\{1^8, 2\}$. Table 2 lists the block multisets for small d .

Table 2: Dimension vectors of finite-dimensional C^* -algebras of complex dimension d , written as multisets of block sizes.

d	Dimension vectors	Count
1	$\{1\}$	1
2	$\{1^2\}$	1
3	$\{1^3\}$	1
4	$\{1^4\}$ and $\{2\}$	2
5	$\{1^5\}$ and $\{1, 2\}$	2
6	$\{1^6\}$ and $\{1^2, 2\}$	2
7	$\{1^7\}$ and $\{1^3, 2\}$	2
8	$\{1^8\}$, $\{1^4, 2\}$, and $\{2^2\}$	3
9	$\{1^9\}$, $\{1^5, 2\}$, $\{1, 2^2\}$, and $\{3\}$	4
10	$\{1^{10}\}$, $\{1^6, 2\}$, $\{1^2, 2^2\}$, and $\{1, 3\}$	4
11	$\{1^{11}\}$, $\{1^7, 2\}$, $\{1^3, 2^2\}$, and $\{1^2, 3\}$	4
12	$\{1^{12}\}$, $\{1^8, 2\}$, $\{1^4, 2^2\}$, $\{2^3\}$, and $\{1^3, 3\}$	5

Remark 3.7 (Recognition). The dimension vector solves the recognition problem for this class as well as the classification problem: given matrix blocks, comparing their size multisets decides $*$ -isomorphism directly, and the comparison is a sort. In the finite-dimensional regime classification, parameterization, and recognition therefore coincide. The classification of finite simple groups separates the three: it delivers the list, partially solves recognition through atlas data, and supplies no uniform parameterization. The separation reappears, in worse form, in section 5. For channels the recognition problem is open in the same way the classification problem is: comparing Choi spectra decides nothing beyond the orbit invariants of theorem 6.8, and no recognition procedure for unitary equivalence of channels is on record.

3.1 States and state spaces

The observable algebra of theorem 3.1 carries state space data that the classification records.

Proposition 3.8 (States of a block algebra). *Let $A = \bigoplus_{i=1}^k M_{n_i}(\mathbb{C})$. Every state ρ on A is of the form*

$$\rho(X_1, \dots, X_k) = \sum_{i=1}^k \text{Tr}(\sigma_i X_i)$$

for positive matrices $\sigma_i \in M_{n_i}(\mathbb{C})$ with $\sum_i \text{Tr} \sigma_i = 1$, and this correspondence is bijective. The pure states of A are the vector states of individual blocks, so the pure state space is the disjoint union of complex projective spaces $\mathbb{CP}^{n_i-1}$.

Proof. A positive functional on a direct sum is blockwise positive, so it suffices to treat one block. On $M_n(\mathbb{C})$ every positive functional is a trace against a positive matrix: the matrix σ with entries $\sigma_{ji} = \rho(E_{ij})$ satisfies $\rho(X) = \text{Tr}(\sigma X)$ for all X by linearity on matrix units, and positivity of ρ gives $\sigma \geq 0$. Normalization of ρ on A reads $\sum_i \text{Tr} \sigma_i = 1$. Purity forces all but one σ_i to vanish and that σ_i to have rank one, which is a vector state of block i . $\square$

The state space of A , as a compact convex body, is therefore determined by the dimension vector: it is the join of the density-matrix bodies of the blocks.

Proposition 3.9 (Automorphisms of block algebras). *Every $*$ -automorphism of $A = \bigoplus_{i=1}^k M_{n_i}(\mathbb{C})$ is the composition of a blockwise inner automorphism and a permutation of blocks of equal size. The dynamics of a finite-dimensional quantum system is therefore a unitary path inside blocks together with block permutations of equal size, and none of it is recorded by the dimension vector beyond the block sizes themselves.*

Proof. By theorem 3.1 every $*$ -automorphism permutes the simple ideals of A , and it can permute only ideals of equal size, being a $*$ -isomorphism. On a single block every $*$ -automorphism of $M_n(\mathbb{C})$ is inner, so after choosing the permutation the automorphism is blockwise conjugation by unitaries. The second statement collects the description. $\square$

Proposition 3.10 (Unique decomposition in the finite-dimensional class). *Decompositions of a finite-dimensional C^* -algebra A into irreducible summands under direct sum exist and are unique up to order and unitary equivalence. The dimension vector is therefore complete data for the decomposition as well as for the isomorphism class, and the uniqueness that fails for interacting systems in general holds in this restricted class.*

Proof. Existence is theorem 3.1. Uniqueness holds because the summands are the simple ideals of the semisimple algebra A , and the minimal nonzero two-sided ideals of a semisimple algebra are unique. Any decomposition into irreducible summands exhibits exactly these ideals, and the order of the summands is not part of the data, matching the multiset form of the dimension vector. $\square$

Proposition 3.11 (Tensor products of block algebras). *Let $A = \bigoplus_i M_{n_i}(\mathbb{C})$ and $B = \bigoplus_j M_{m_j}(\mathbb{C})$. Then $A \otimes B \cong \bigoplus_{i,j} M_{n_i m_j}(\mathbb{C})$, and the dimension vector of $A \otimes B$ is the multiset of products $\{n_i m_j\}$. In particular the dimension vector of a tensor product is computed from the factor dimension vectors without reference to any presentation.*

Proof. The tensor product of matrix algebras satisfies $M_n(\mathbb{C}) \otimes M_m(\mathbb{C}) \cong M_{nm}(\mathbb{C})$ as C^* -algebras, and the tensor product of direct sums distributes: $A \otimes B \cong \bigoplus_{i,j} M_{n_i}(\mathbb{C}) \otimes M_{m_j}(\mathbb{C})$. Theorem 3.1 then identifies the result blockwise. $\square$

4 Operator algebras and factors

Infinite-dimensional quantum systems require operator-algebraic observables. The classification of von Neumann algebras reduces to factors, and the classification of factors is complete on the amenable side and out of reach of any finite list of types on the general side.

Definition 4.1 (Factor types). A factor M is of type I if it contains minimal projections, of type II_1 if it is infinite-dimensional with a finite faithful normal trace, of type II_∞ if it is properly infinite with a semifinite trace, and of type III if it admits no nonzero finite trace. The type III factors are stratified by the Connes spectrum of the modular automorphism group, with parameters $\lambda \in (0, 1)$ and the two endpoint cases III_0 and III_1 .

Murray-von Neumann comparison theory is the tool behind the trichotomy. Two projections p and q of a von Neumann algebra are equivalent when a partial isometry u with $u^*u = p$ and $uu^* = q$ exists, and in a factor any two projections are comparable: either p is equivalent to a subprojection of q or conversely. Type I factors contain minimal projections, type II factors contain no minimal projections but do contain nonzero finite projections, and type III factors have every nonzero projection infinite. The type of a factor is therefore the first invariant of theorem 4.4.

The reduction to factors is the operator-algebraic analogue of block decomposition.

Definition 4.2 (Amenable factors). All factors in this section have separable predual. A factor is hyperfinite, or approximately finite-dimensional, when it is the norm closure of an increasing union of finite-dimensional subalgebras, and injective when a norm-one projection of $B(H)$ onto it exists. For factors with separable predual the two conditions coincide, and a factor satisfying them is amenable. The unique amenable factor of type II_1 with separable predual is the hyperfinite factor R .

Theorem 4.3 (Central decomposition). *Every von Neumann algebra M is a direct integral of factors over a measure space, unique up to measure-class isomorphism of the base. The isomorphism classification of von Neumann algebras therefore reduces to the classification of factors together with the measure-theoretic data of the decomposition.*

Theorem 4.3 is the Murray-von Neumann direct-integral decomposition; we refer to the original work and to the standard accounts for the proof. The classification content sits in the factors.

Theorem 4.4 (Classification of amenable factors). *Among factors with separable predual the amenable (equivalently injective, equivalently hyperfinite) factors are classified as follows. For each $n \geq 1$ there is a unique factor of type I_n and a unique factor of type I_∞ . There is a unique amenable factor of type II_1 , the hyperfinite factor, and a unique amenable factor of type II_∞ . For each $\lambda \in (0, 1)$ there is a unique amenable factor of type III_λ , and a unique amenable factor of type III_1 . Amenable factors of type III_0 are classified by the ergodic flow of weights. The invariants in force are the Murray-von Neumann type together with the Connes invariants. Separable predual is essential: among non-separable injective II_1 factors uniqueness fails, and there are 2^c pairwise non-isomorphic ones.*

Status of the proof. The type I and type II cases are Murray-von Neumann's, together with the existence and uniqueness of the hyperfinite II_1 factor. The classification of injective factors, including the type III_λ cases for $\lambda \neq 0$, is Connes's theorem of 1976, building on his 1973 classification of type III factors by the invariants of theorem 4.1. Uniqueness of the injective factor of type III_1 was completed by Haagerup in 1987 through the bicentralizer problem. The statements are established theorems of the literature; the proofs are deep analytic arguments in von Neumann algebra theory and are not reproduced here. What is proved in this paper is the finite-dimensional content of section 3 and the channel content of section 6, each in full. $\square$

4.1 The hyperfinite factor

In type II_1 the hyperfinite factor R is the norm closure of an increasing union of full matrix algebras $M_{2^k}(\mathbb{C})$ along the embeddings $A \mapsto A \otimes I_2$, and every amenable II_1 factor with separable predual is isomorphic to R . The type III_λ factors for $\lambda \in (0, 1)$ arise differently: as infinite tensor products of finite type I factors carrying non-tracial product states (the Powers factors), not as inductive limits of matrix algebras along unital embeddings. The unique amenable factor of type III_1 is the limit case completed by Haagerup's theorem.

Within type III_0 the amenable factors are classified by the ergodic flow of weights up to conjugacy of the flow, and this is the one stratum whose invariant is an ergodic object rather than a number.

The factor R is the reference object of the classification. It lies inside the classified regime of theorem 4.4, it is the unique separable injective II_1 factor, and the uncountable family of theorem 5.1 is produced by varying the central-sequence structure of algebras built from it. The boundary between the amenable side and the general side is therefore a boundary inside the separable II_1 factors themselves.

Remark 4.5 (Type III strata). The Connes spectrum $S(M) = \bigcap_\varphi \text{Sp}(\Delta_\varphi)$ is the intersection over normal faithful states φ of the spectra of the modular operators Δ_φ , a subset of $[0, \infty)$, and the flow of weights is the ergodic flow attached to the modular automorphism group through the central decomposition. The stratum data are read off from $S(M) \cap \mathbb{R}_+^*$: it is $\{1\}$ in type III_0 , $\lambda^\mathbb{Z}$ in type III_λ for $\lambda \in (0, 1)$, and $(0, \infty)$ in type III_1 . Theorem 4.4 classifies the amenable factors through these invariants and the ergodic flow of weights in type III_0 . Beyond the amenable class the strata are coarse: fundamental groups of type III factors vary widely, and the structure theory past the Connes spectrum is open. No completeness claim for non-amenable type III factors is used anywhere in this paper.

Remark 4.6 (The Elliott program). For separable simple nuclear C^* -algebras the analogous program classifies algebras by K -theoretic and tracial data. The program is unfinished. Toms constructed a simple unital algebra shape equivalent to its tensor product with an infinite-dimensional UHF algebra and with the same tracial simplex, yet not isomorphic to it, so K -theory and traces together do not complete the classification without further hypotheses. This paper draws no classification conclusion from the Elliott program.

5 The McDuff obstruction

The most consequential theorem for the classification program is negative.

Theorem 5.1 (McDuff 1969). *There exist uncountably many pairwise non-isomorphic separable type II_1 factors.*

Consequence for classification. The theorem asserts continuum many isomorphism classes: the separable II_1 factors form an uncountable family with no repetitions up to isomorphism. Two consequences follow. No finite or countable list of factor types can exhaust the family, so a CFSG-style list of types cannot be complete outside a restricted class, and any classification theorem for II_1 factors must restrict its class (for instance to the amenable factors of theorem 4.4) or weaken its conclusion to a coarser equivalence. The theorem by itself does not decide whether a classification by continuous parameters exists; that question belongs to the descriptive set theory of von Neumann algebra isomorphism and is not settled here. What the theorem does settle is that the separation is visible in the operator-algebraic structure itself: the construction separates factors through their central sequences. $\square$

Corollary 5.2 (No finite list of factor types). *No finite or countable list of pairwise non-isomorphic factor types exhausts the separable II_1 factors, and no CFSG-style list of factor types is complete outside a restricted class such as the amenable factors.*

Proof. A finite or countable list covers countably many isomorphism classes, and theorem 5.1 supplies continuum many. The second statement is the first applied to the CFSG shape of a classification theorem. $\square$

Remark 5.3 (Honest record). Theorem 5.1 marks the boundary of the classifiable territory: it is a classification result of the same standing as the positive theorems, not a difficulty to be engineered around. The amenable factors lie inside the boundary by theorem 4.4, and the general II_1 factors lie outside it.

Inside the amenable class the classification has the shape of a CFSG-style theorem: a list of types, a complete set of invariants, and a uniqueness statement for each type. The obstruction says this shape cannot extend to all separable II_1 factors, so the shape must be sought in restricted classes or with weakened equivalences. The amenable classification is one instance of the pattern; whether rigidity hypotheses on non-amenable factors define another instance is one of the open problems of section 9.

6 Channels and dilations

6.1 Choi-Kraus normal form

The morphisms of finite-dimensional quantum systems that appear in measurement and evolution are channels (theorem 2.4). Their normal form is the first of the two canonical-form theorems of the paper.

Lemma 6.1 (Choi correspondence). *For a linear map $\Phi : M_n(\mathbb{C}) \rightarrow M_m(\mathbb{C})$ with Choi matrix $J(\Phi) = \sum_{i,j} E_{ij} \otimes \Phi(E_{ij})$ the following hold: Φ is completely positive if and only if $J(\Phi) \geq 0$, and Φ is trace preserving if and only if $\text{Tr}_m J(\Phi) = I_n$, where Tr_m is the partial trace over the second tensor factor.*

Proof. Complete positivity is tested on positive elements of $M_n(\mathbb{C}) \otimes M_k(\mathbb{C})$ for all k , and for $k = n$ the condition $(\text{id} \otimes \Phi)(P) \geq 0$ on rank-one projections $P = |\psi\rangle\langle\psi|$ with $|\psi\rangle = \sum_i e_i \otimes e_i$ reads $J(\Phi) \geq 0$; the general case follows by direct sums. For the trace condition, $\text{Tr}_m J(\Phi)$ has (i, j) entry $\text{Tr}(\Phi(E_{ij}))$, and Φ preserves trace exactly when $\text{Tr}(\Phi(E_{ij})) = \delta_{ij}$, which is $\text{Tr}_m J(\Phi) = I_n$. $\square$

Example 6.2 (A family of one Kraus rank). For $p \in (0, 1)$ set $A_0 = \sqrt{1-p} I_2$ and $A_i = \sqrt{p/3} \sigma_i$ for the three Pauli matrices σ_i . The map $\Phi_p(X) = \sum_{k=0}^3 A_k X A_k^*$ satisfies $\sum_k A_k^* A_k = I_2$, so it is a qubit channel. Its Choi matrix has eigenvalues $2(1-p)$ and $2p/3$ with multiplicity three, all positive, so the Kraus rank is 4 for every $p \in (0, 1)$. The Choi spectrum varies with p , and unitary equivalence of channels preserves the Choi spectrum, so the family consists of pairwise unitarily inequivalent channels sharing one Kraus rank.

Example 6.3 (Choi matrix of a qubit channel). For the channel Φ_p of theorem 6.2 the Kraus vectors are the vectorizations of I_2 and of the Pauli matrices, which are orthogonal for the Hilbert-Schmidt pairing. The Choi matrix is therefore diagonal in the basis these vectors span, with eigenvalues $2(1-p)$ and $2p/3$ with multiplicity three. Trace preservation reads $\text{Tr}_m J(\Phi_p) = I_2$ on these eigenvalues, and complete positivity reads $J(\Phi_p) \geq 0$, which holds for $p \in (0, 1)$. The eigenvalue list is the input of the minimization statement of theorem 6.8.

Theorem 6.4 (Choi-Kraus normal form). *Let $\Phi : M_n(\mathbb{C}) \rightarrow M_m(\mathbb{C})$ be linear. Then Φ is completely positive if and only if there are matrices $A_1, \dots, A_r \in M_{m \times n}(\mathbb{C})$ with*

$$\Phi(X) = \sum_{i=1}^r A_i X A_i^* \quad \text{for all } X \in M_n(\mathbb{C}),$$

and one may take r equal to the rank of the Choi matrix $J(\Phi) = \sum_{i,j} E_{ij} \otimes \Phi(E_{ij})$. The minimal such r is the Kraus rank of Φ and equals the rank of $J(\Phi)$.

Proof. Write $J(\Phi) \in M_{nm}(\mathbb{C})$ in block form as the $n \times n$ block matrix with blocks $\Phi(E_{ij})$. Complete positivity of Φ is equivalent to positivity of $J(\Phi)$, and positivity is equivalent to a factorization $J(\Phi) = \sum_{k=1}^r v_k v_k^*$ with $r = \text{rank } J(\Phi)$ and vectors $v_k \in \mathbb{C}^n \otimes \mathbb{C}^m$. Fix an orthonormal basis $e_1, \dots, e_n$ of $\mathbb{C}^n$ and decompose each $v_k = \sum_{j=1}^n e_j \otimes w_{kj}$ with $w_{kj} \in \mathbb{C}^m$. Let $A_k \in M_{m \times n}(\mathbb{C})$ be the matrix whose j -th column is w_{kj} . The (i, j) block of $J(\Phi)$ is then $\sum_{k=1}^r w_{ki} w_{kj}^* = \sum_{k=1}^r A_k E_{ij} A_k^*$, so $\Phi(E_{ij}) = \sum_k A_k E_{ij} A_k^*$ for all i, j and by linearity $\Phi(X) = \sum_k A_k X A_k^*$ for all X . Conversely any such representation writes $J(\Phi)$ as a sum of s rank-one positive terms, so $s \geq \text{rank } J(\Phi)$ and the minimal number of Kraus operators equals r . $\square$

Remark 6.5 (Normal form versus classification). Theorem 6.4 is a normal-form theorem, not a classification of channels up to isomorphism. The Kraus rank is an invariant of the minimal presentation of one channel. Section 6 exhibits channels with equal Kraus rank that no unitary change of presentation identifies, and the literature records no isomorphism classification of channels beyond the normal form.

6.2 Stinespring dilations

Stinespring's theorem gives every channel a canonical dilation through an environment, and a minimal dilation is unique up to unitary. The statement is the second canonical-form theorem of the paper.

Theorem 6.6 (Stinespring dilation). *Let $\Phi : M_n(\mathbb{C}) \rightarrow M_m(\mathbb{C})$ be a channel. Then Φ admits a Stinespring dilation (K, V) in the sense of theorem 2.4, and minimal dilations exist. Any two minimal dilations (K, V) and (K', V') of Φ are unitarily equivalent: there is a unitary $U : K \rightarrow K'$ with $(I_m \otimes U)V = V'$. The dimension $\dim K$ of a minimal dilation equals the Kraus rank of Φ .*

Proof. Take a Kraus presentation $\Phi(X) = \sum_{k=1}^r A_k X A_k^*$ as in theorem 6.4, with r the Choi rank. Set $K = \mathbb{C}^r$ and define $V : \mathbb{C}^n \rightarrow \mathbb{C}^m \otimes K$ by $V\psi = \sum_{k=1}^r (A_k \psi) \otimes e_k$ for the standard basis e_k of K . Then $V^*V = \sum_k A_k^* A_k = I_n$ because Φ preserves trace, so V is an isometry, and $\text{Tr}_K(VXV^*) = \sum_k A_k X A_k^* = \Phi(X)$. A dilation on a space of dimension less than r would yield a Kraus presentation with fewer than r operators, which theorem 6.4 forbids, so this dilation is minimal. For uniqueness, both K and K' are spanned by vectors of the form $(w^* \otimes I)V\psi$ and $(w^* \otimes I)V'\psi$ with $w \in \mathbb{C}^m$ and $\psi \in \mathbb{C}^n$, the correspondence between these vectors preserves inner products because both dilations realize the same map Φ , and it extends to a unitary $U : K \rightarrow K'$ with $(I_m \otimes U)V = V'$. For completely positive unital maps on a general C^* -algebra the dual picture holds: $\Phi(a) = V^*\pi(a)V$ with π a unital $*$ -representation and V an isometry, which is Stinespring's original form. $\square$

Remark 6.7 (Two pictures). Channels appear in two standard forms. In the Schrödinger picture a channel is completely positive and trace preserving, acting on states and dilating through an environment as in theorem 2.4. In the Heisenberg picture a channel is a unital completely positive

map of observable algebras, and its Stinespring form is the compression $\Phi(a) = V^*\pi(a)V$ named at the end of theorem 6.6. The two pictures are related by adjunction, the dilation statements hold in either, and the invariant content is the same.

For channels on matrix algebras the minimal dilation is finite-dimensional and the unitary freedom of theorem 6.6 is exactly the change-of-presentation freedom of Kraus operators.

Proposition 6.8 (Kraus rank is a normal-form invariant). *Let $\Phi : M_n(\mathbb{C}) \rightarrow M_m(\mathbb{C})$ be a channel. The Kraus rank of Φ equals the rank of its Choi matrix, equals the minimal number of Kraus operators in any representation of Φ , and equals the dimension of the environment K in a minimal Stinespring dilation. The Kraus rank is preserved by unitary changes of Kraus presentation and by composition of Φ with unitary conjugations of domain and codomain. It is not a complete invariant of channels: two channels with equal Kraus rank need not be unitarily equivalent.*

Proof. The equalities with the Choi rank and the minimal Kraus number are theorem 6.4; the equality with the minimal dilation dimension is the construction of theorem 6.6 specialized to matrix algebras. Unitary conjugation of the Kraus operators replaces each A_i by WA_iU^* with U, W unitary and does not change the rank of the Choi matrix. For the negative statement take the channel family Φ_p of theorem 6.2. Its Kraus rank is 4 for every $p \in (0, 1)$, its Choi eigenvalues $2(1 - p)$ and $2p/3$ vary with p , and unitary equivalence of channels preserves the Choi spectrum. Distinct values of p give inequivalent channels with equal Kraus rank. $\square$

Remark 6.9 (Prohibition recorded). Theorem 6.8 records the exact boundary. Kraus rank may be used to compare normal forms of one channel and to detect minimality of a dilation. It may not be used as an isomorphism classification of channels, and no theorem of this paper treats it as one. The Kraus rank is an invariant of the minimal dilation layer of structure and carries no classification claim for the channel layer.

Proposition 6.10 (Composition of channels). *Let $\Phi : M_n(\mathbb{C}) \rightarrow M_m(\mathbb{C})$ and $\Psi : M_m(\mathbb{C}) \rightarrow M_p(\mathbb{C})$ be linear. Then*

$$J(\Psi \circ \Phi)_{(i\alpha), (j\beta)} = \sum_{k,l=1}^m J(\Phi)_{(ik), (jl)} J(\Psi)_{(k\alpha), (l\beta)},$$

with composite indices as in theorem 6.1. The right-hand side is the link product of $J(\Phi)$ and $J(\Psi)$: matrix multiplication of $J(\Phi)$ after partial transpose on the shared subsystem, contracted with $J(\Psi)$ along that subsystem. Composition of channels is therefore a link-product contraction, and the Choi matrix of a composite is determined by the Choi matrices of the factors.

Proof. Write $\Phi(E_{ij}) = \sum_{k,l} c_{kl}^{ij} E_{kl}$ and read $J(\Phi)_{(ik), (jl)} = c_{kl}^{ij}$ and $J(\Psi)_{(k\alpha), (l\beta)} = \Psi(E_{kl})_{\alpha\beta}$. The right-hand side of the claimed identity is then $\sum_{k,l} c_{kl}^{ij} \Psi(E_{kl})_{\alpha\beta} = \Psi(\Phi(E_{ij}))_{\alpha\beta}$, which is $J(\Psi \circ \Phi)_{(i\alpha), (j\beta)}$ by definition. $\square$

Remark 6.11 (What a channel classification would need). A classification of channels up to unitary equivalence would need invariants separating the orbits of the unitary action $\{A_k\} \mapsto \{WA_kU^*\}$ on Kraus presentations. The Choi spectrum is one such invariant and is not complete, as theorem 6.2 shows at the level of spectra alone; the orbit data of a presentation involves the arrangement of its Kraus vectors in $\mathbb{C}^n \otimes \mathbb{C}^m$ beyond their Gram matrix. No complete set of orbit invariants is on record, and this is the fourth open problem of section 9.

7 Tensor categories and superselection

Superselection sectors of a quantum field theory organize into tensor categories, and two reconstruction theorems classify such categories under explicit hypotheses. The hypotheses are load-bearing: both theorems fail or become open when they are dropped, and one equivalence-negligible layer turns out to carry the group.

Definition 7.1 (Superselection category). Fix a local net of observable algebras satisfying the Haag-Kastler axioms and the DHR selection criterion: representations that are equivalent to the vacuum representation on the algebra of every causally disconnected region. Under Haag duality these representations correspond to localized transportable endomorphisms of the observable algebra, and their intertwiners form a symmetric tensor $*$ -category with conjugates, direct sums, and subobjects.

The tensor structure of that category is concrete. Composition of endomorphisms supplies the tensor product, intertwiners supply the morphisms, conjugate endomorphisms supply duals, and the statistics of composing an endomorphism with itself distinguishes bosonic from fermionic sectors. Haag duality is the step that passes from representations to endomorphisms, and the selection criterion is what bounds the sectors; modified selection rules change the category.

Theorem 7.2 (Doplicher-Roberts reconstruction). *Let $\mathcal{C}$ be a symmetric tensor $*$ -category with conjugates, direct sums, and subobjects, and with $\text{End}(\mathbf{1}) = \mathbb{C}$. Then $\mathcal{C}$ is equivalent, as a symmetric tensor $*$ -category, to the category of finite-dimensional unitary representations of a compact supergroup G , and G is unique up to isomorphism.*

Theorem 7.2 is stated with its hypotheses on the face of the statement. The symmetric structure, the existence of conjugates, and the scalar endomorphism ring of the unit are each used in the proof; a category missing them lies outside the theorem. The reconstruction of the gauge group from superselection data is the completion of the Doplicher-Haag-Roberts analysis, and the supergroup grading accounts for fermionic sectors.

Remark 7.3 (Selection criteria matter). The DHR criterion selects representations with trivial behaviour at infinity. Long-range forces such as electromagnetism violate the criterion, and the resulting categories fall outside theorem 7.2 unless the analysis is redone with modified selection rules. A classification claim for superselection categories is therefore always relative to the selection criterion in force.

Remark 7.4 (What reconstruction does not do). The Doplicher-Roberts theorem recovers a symmetry group from sector data. It does not recover spacetime, dynamics, or states, and it does not classify the systems whose sectors are being read.

Theorem 7.5 (Deligne). *Let $\mathcal{C}$ be a $\mathbb{C}$ -linear, abelian, rigid symmetric tensor category over $\mathbb{C}$ of subexponential growth, with $\text{End}(\mathbf{1}) = \mathbb{C}$. Then $\mathcal{C}$ is equivalent, as a symmetric tensor category, to the category of finite-dimensional representations of an affine supergroup scheme. In particular the category is Tannakian in the super sense.*

Deligne's theorem classifies a purely categorical hypothesis class, without reference to any net of observables. The two reconstruction theorems differ in their inputs and agree in their outputs. The Doplicher-Roberts theorem starts from the sectors of a net of observables, so its hypotheses are analytic as well as categorical: Haag duality, the selection criterion, and the tensor structure of intertwiners. Deligne's theorem starts from $\mathbb{C}$ -linearity, abelianness, rigidity, symmetry, and subexponential growth. Both conclusions are representation categories of supergroups, so the two theorems are the analytic and the categorical face of one reconstruction pattern, and neither

classifies the systems whose sectors or categories are being read. Together theorems 7.2 and 7.5 settle the classification of the symmetric categories that appear as superselection data, and of the finite-growth categories that appear in low-dimensional and combinatorial models. The case of fusion categories (finitely many simple objects, semisimple, finite categorical dimension) is open beyond these hypotheses.

Proposition 7.6 (Etingof-Gelaki counterexample). *There exist finite non-isomorphic groups G_1 and G_2 whose representation categories over $\mathbb{C}$ are equivalent as tensor categories, but not as symmetric tensor categories. Consequently an equivalence of tensor categories that forgets the symmetric structure cannot recover the group, and any classification of such categories built on that equivalence is false.*

Proof. Etingof and Gelaki exhibit the affine symplectic group of a vector space over the field of two elements and a corresponding affine pseudosymplectic group: these are non-isomorphic and isocategorical, meaning their representation categories are monoidally equivalent. The symmetric structure of the two categories differs, and Tannaka reconstruction for symmetric tensor categories recovers the group from the symmetric structure alone. The symmetric layer is therefore load-bearing in the sense of theorem 2.6, and an equivalence that discards it identifies objects that the classification must separate. $\square$

Remark 7.7 (Equivalence discipline). Any equivalence of tensor categories must declare the layers it preserves. The standing qualification on relational characterizations applies in full: “Yoneda does not, by itself, establish physical equivalence, empirical adequacy, or a classification theorem.”

Remark 7.8 (Fusion categories). Every fusion category is the representation category of a weak Hopf algebra, a result obtained through the module-category machinery of Ostrik, and this gives a presentation theorem rather than a classification. The structural conjectures of Etingof, Nikshych, and Ostrik, in particular the existence of a coherent monoidal isomorphism $A \cong A^{**}$ for every fusion category A , remain open, and this paper states all fusion-category claims as conditional on those conjectures.

8 Monotone metrics on quantum states

The measure-theoretic family contributes the third invariant type of the paper, and the quantum case is where a classical uniqueness theorem becomes a parametric classification.

Definition 8.1 (Monotone metric). Let $\mathcal{D}_n$ denote the manifold of strictly positive density matrices of $M_n(\mathbb{C})$, an open subset of the affine space of self-adjoint trace-one matrices, with tangent space $T_\rho \mathcal{D}_n$ the space of self-adjoint traceless matrices at ρ . A monotone metric assigns to each $\rho \in \mathcal{D}_n$ an inner product on $T_\rho \mathcal{D}_n$ such that the metric decreases under every completely positive trace-preserving map. In the classical setting the corresponding notion is monotonicity under stochastic maps.

Remark 8.2 (Classical uniqueness). Chentsov’s theorem classifies monotone metrics on the simplex of probability distributions: the Fisher metric is the unique metric, up to a positive constant, that decreases under every stochastic map. The quantum statement of theorem 8.3 replaces uniqueness by a parametric family, and no rescaling recovers a unique quantum metric. The contrast is the quantum instance of a classical-to-quantum gap that runs through the series: uniqueness statements on the classical side do not transfer.

Theorem 8.3 (Petz). *Monotone metrics on $\mathcal{D}_n$ are in bijection with operator monotone functions $f : (0, \infty) \rightarrow (0, \infty)$ that satisfy the symmetry condition $f(t) = tf(1/t)$ and the normalization $f(1) = 1$. Given such a function f , the corresponding metric on the tangent space $T_\rho \mathcal{D}_n$ is*

$$g_\rho^f(A, B) = \text{Tr}\left(A^*[R_\rho f(L_\rho R_\rho^{-1})]^{-1}(B)\right),$$

where L_ρ and R_ρ are the left and right multiplication operators by ρ on $M_n(\mathbb{C})$. Distinct normalized operator monotone functions give distinct monotone metrics.

Status of the proof. The classification is Petz's theorem. The condition $f(t) = tf(1/t)$ is the symmetry of the resulting form under exchange of A and B , and the normalization $f(1) = 1$ fixes the scale: without it every positive scalar multiple of f satisfies the symmetry condition and rescales the metric, and the correspondence is not a bijection. Distinct normalized functions give distinct metrics already on 2×2 density matrices. The proof is an operator-geometric argument in monotone matrix theory and is not reproduced here. The parametric conclusion is recorded in the sources of Petz and of Petz-Sudar, who construct and compare explicit members of the family and state the contrast with Chentsov uniqueness outright. $\square$

Corollary 8.4 (Parametricity of the quantum invariant). *The monotone metrics of quantum states are classified by a continuous parameter (the normalized operator monotone function), not by a discrete choice. There is no quantum analogue of Chentsov uniqueness, and any classification statement for quantum monotone metrics must accept a parametric invariant in place of a unique object.*

Example 8.5 (Two standard metrics). The function $f(t) = (1 + t)/2$ is operator monotone and satisfies $f(1) = 1$ and $f(t) = tf(1/t)$. In the eigenbasis of ρ with eigenvalues λ_i its metric reads $g_\rho(A, A) = \sum_{i,j} 2|A'_{ij}|^2/(\lambda_i + \lambda_j)$ with $A' = U^*AU$, which is the metric of the symmetric logarithmic derivative. The function $f(t) = (1 + \sqrt{t})^2/4$ again satisfies both conditions and yields the Wigner-Yanase metric. The two metrics coincide on tangent directions commuting with ρ and differ on every self-adjoint traceless direction with an off-diagonal entry in the eigenbasis of ρ , which is theorem 8.4 in concrete form.

Remark 8.6 (Distinguishability). A monotone metric measures how fast nearby states separate under measurements, and monotonicity makes the separation decrease under every channel. Choosing a metric amounts to choosing a convention for statistical distance on quantum states, and theorem 8.3 shows the choice is a continuous parameter. Any classification or equivalence statement that uses statistical distance on quantum states must carry the parameter of theorem 8.4 or fix it explicitly.

9 Classification regimes

Three regimes hold among the quantum classification problems: complete classification under stated hypotheses, uncountably many classes admitting no finite list of types, and open problems. Table 3 records each domain with its invariant and source theorem.

The regimes show a pattern the rest of the series exploits. Classification is complete exactly where the invariant is discrete or parametric and the hypotheses are stated; finite lists stop where the structure theory manufactures uncountably many non-isomorphic objects; and the question is open where a presentation theorem exists but the invariants do not separate. The finite-dimensional theory and the amenable theory differ in technique but agree in shape: both reduce a system to a

Table 3: Classification regimes for quantum systems, with the invariant in force and the standing of each row.

Domain	Invariant	Standing	Source
Finite-dimensional C^* -algebras	Dimension vector	Classified completely	Theorem 3.1
Completely positive maps	Kraus rank and Choi matrix	Normal form only	Theorem 6.4
Minimal dilations	Unitary class of the dilation	Classified completely	Theorem 6.6
Amenable factors	Murray-von Neumann type and Connes invariants	Classified completely	Theorem 4.4
General II_1 factors	Continuum many isomorphism classes	No finite list of types	Theorem 5.1
Symmetric tensor categories under the hypotheses of theorem 7.5	Underlying supergroup	Classified completely	Theorems 7.2 and 7.5
Tensor equivalence without symmetric structure	None can recover the group	Proved insufficient	Theorem 7.6
Monotone metrics on quantum states	Operator monotone function	Parametric classification	Theorem 8.3
Fusion categories	Weak Hopf presentation	Open	Theorem 7.8
Non-amenable type III factors	Connes strata only	Open	Theorem 4.5
Channels up to isomorphism	Kraus rank is not complete	Open	Theorem 6.9

list of standard pieces plus invariant data. The McDuff obstruction marks where that shape cannot persist.

Five kinds of classification result appear in table 3. The dimension-vector classification is complete and parameterized, in the manner of the Dynkin classification of semisimple Lie algebras. The amenable factor classification is complete and analytic, its invariants computed from operator-algebraic structure rather than read off combinatorial data. The monotone-metric classification is parametric, in the manner of a classification by continuous parameters.

The McDuff corollary is a complete obstruction: its classification content is the proof that no finite list exists. The remaining rows are open problems with presentation theorems but no separating invariants. These five kinds separate cleanly, and a quantum classification claim that mixes two of them is misstated.

Five open problems bound the current frontier.

1. The Elliott program for separable simple nuclear C^* -algebras lacks a finished classification statement with its universal coefficient theorem resolved.
2. The structural conjectures for fusion categories, beginning with coherent double-dual isomorphisms, are unproved, and every fusion-category claim in this paper is conditional on them.
3. The structure of non-amenable type III factors beyond the Connes strata is open.

4. No isomorphism classification of quantum channels exists beyond the Stinespring normal form, and theorem 6.8 shows that the obvious invariant does not provide one.
5. Whether any restricted classification of non-amenable II_1 factors survives under rigidity hypotheses is not addressed by the sources consulted here and is recorded as open.

10 Computational evidence

The source files that accompany this paper as supplementary material contain executable models and three experiment scripts; they live in the directory `src/vol-06-quantum-systems/`. They are computational evidence for the finite-dimensional and channel statements. They are not proofs, and no part of the paper rests on them for validity.

The Haskell module `Instances.hs` instantiates the taxonomy classes for this paper’s physical systems: finite-dimensional block algebras, Hilbert-space systems with observable algebras, operator-algebra systems with factor data, quantum channels with Kraus data, and superselection categories with tensor data. The instances cover the system-record class, the Hilbert-space and operator-algebra family class, the probability and measure family class on the quantum side, the tensor-category family class, the construction and decomposition hooks, irreducibility of matrix blocks under direct sum, and mathematical equivalence of block algebras by dimension vector. The instances name their methods after the definitions and theorem targets of this paper (dimension vector, minimal Kraus rank, Stinespring minimal dilation, operator monotone parameter). These are executable models of the taxonomy, validated against the statements of the paper; no machine-checked formalization is claimed for them.

Three Python scripts carry the finite-dimensional experiments. The first enumerates small finite-dimensional C^* -algebras by dimension vector and checks the counting statement of theorem 3.6 and the isomorphism criterion of theorem 3.1. The second constructs Choi matrices of sample channels, computes Kraus ranks, minimizes Kraus presentations, and exhibits channels of equal Kraus rank that no unitary identification matches, evidencing theorem 6.8. The third builds monotone metrics from normalized operator monotone functions on 2×2 density matrices, checks positivity of the resulting forms, and confirms that distinct functions give distinct metrics, evidencing theorem 8.3 and theorem 8.4. Each script prints its checks and exits nonzero on failure. Table 4 records the checks of a recorded run.

Appendix A maps every computational claim of the paper to the file and function that evidences it.

11 Discussion

Three complete classifications cover finite-dimensional observable algebras, amenable factors, and symmetric tensor categories under stated hypotheses. Finite-dimensional observable algebras are classified by dimension vector with a self-contained proof. Amenable factors are classified by Murray-von Neumann type and Connes invariants with proofs in the literature of Murray-von Neumann, Connes, and Haagerup. Symmetric tensor categories under the hypotheses of Deligne and Doplicher-Roberts are classified by the underlying supergroup. Beyond these, the general II_1 factors admit no finite list of types, and the classification question changes shape there from listing objects to delimiting the classifiable classes.

Three limitations bound the results. The first is exactness of equivalence. Every classification here is up to an equivalence that preserves named layers of structure, and theorem 7.6 shows that

Table 4: Checks and recorded outputs of the three experiment scripts in a single run.

Script	Check	Recorded output
<code>finite_dim_cstar_decomposition.py</code>	Isomorphism criterion on all algebras up to dimension 16	No failures
<code>finite_dim_cstar_decomposition.py</code>	Algebra counts per dimension	Matches table 2 through $d = 12$
<code>finite_dim_cstar_decomposition.py</code>	Tensor products multiply block sizes	$\{2, 3\} \otimes \{2\} = \{4, 6\}$
<code>stinespring_kraus_rank.py</code>	Choi rank of the channel family of theorem 6.2	Rank 4 at all sampled p
<code>stinespring_kraus_rank.py</code>	Minimal Kraus presentation preserves the channel	Four operators, channel preserved
<code>stinespring_kraus_rank.py</code>	Unitary invariance of Kraus rank	Rank preserved
<code>petz_monotone_metrics.py</code>	Positivity of Petz forms on sampled tangent vectors	Minimum sampled value 0.5755
<code>petz_monotone_metrics.py</code>	Distinct functions give distinct metrics	Distinct on all samples

dropping the symmetric layer changes the answer. The classification of mathematical representations is therefore always relative to the preserved structure, and none of it establishes physical equivalence of the systems the representations describe. The second limitation is hypothesis dependence. Theorems 7.2 and 7.5 hold under conjugates, direct sums, subobjects, scalar endomorphisms of the unit, and in Deligne’s case $\mathbb{C}$ -linearity, abelianness, rigidity, and subexponential growth in characteristic zero; relaxed selection criteria or growth hypotheses leave the classified regime. The third limitation is depth of the record. The finite-dimensional proofs are given in full here; the operator-algebraic and categorical proofs are established in the sources cited and are not reproduced, and the claim on each is only as strong as those sources.

The parametric result of section 8 deserves separate emphasis because it constrains future work. Quantum monotone metrics are classified by a continuous parameter, so any proposed uniqueness statement for quantum information geometry must either fix the parameter or weaken to the parametric family. Extending classical uniqueness statements across the classical-to-quantum boundary is not a defensible move.

The three complete classifications instantiate the three structural families of the quantum domain with their classifying invariants in place: Hilbert spaces and operator algebras through the dimension vector and the Murray-von Neumann type with Connes invariants, probability and measure structure through the operator monotone parameter, and tensor categories through the underlying supergroup of a symmetric tensor category. The McDuff obstruction joins the register of non-classification theorems, and the Etingof-Gelaki counterexample fixes the equivalence discipline for any later classification by tensor data. The evidence for the program’s target statement is therefore mixed: restricted-class theorems exist in quantum theory, and obstruction theorems mark exactly where they stop.

No theorem here states that any two of the structural families used above are logically independent, and no theorem here establishes physical or empirical equivalence of any two systems. The long-term classification target of the program remains a research objective; the results of this paper are consistent with it in the classified islands and constrain it by theorem in the non-classifiable ocean. The open problems of section 9 are the natural restricted-class targets next, and each has the shape of a solved problem: a named class, a declared equivalence, and an invariant that either separates or provably cannot.

A complete quantum classification program would need three further deliverables, and this paper supplies none of them. It would need a normal form for channels that upgrades to a classification under a named equivalence, a structure theory for fusion categories that decides the double-dual coherence question, and a restricted classification of non-amenable factors under explicit rigidity hypotheses. Each deliverable is a classification problem of the same shape as the solved ones, and the obstructions recorded here bound what any solution can claim.

12 Conclusion

Quantum physical systems admit classification by dimension vector in finite dimension, by Murray-von Neumann type and Connes invariants on the amenable side of operator algebra, and by underlying supergroup in the symmetric tensor categories of superselection theory. Completely positive maps admit a minimal Stinespring normal form whose Kraus rank is not a channel classification. Quantum monotone metrics are parametric, classified by operator monotone functions rather than by a unique metric. Against these classifications stands the McDuff obstruction: uncountably many separable II_1 factors exist, so no finite list of factor types is exhaustive and no CFSG-style list exists outside restricted classes. The open problems of the Elliott program, fusion-category structure, non-amenable type III strata, and channel isomorphism define the current frontier. The supplementary code records the finite-dimensional content as computational evidence and nothing more.

The negative results are part of the answer. The McDuff obstruction and the Etingof-Gelaki counterexample say where classification stops and which layers of structure carry it, and a classification program that records only its successes would misreport the state of the subject.

A Code evidence

Table 5 maps each computational claim of the paper to the file and function that evidences it in `src/vol-06-quantum-systems/`.

The experiments reproduce by running each script under `src/vol-06-quantum-systems/` with Python 3, and the taxonomy layer compiles against the shared classes in `src/Taxonomy/`.

References

- [1] G. J. Murphy, *C*-Algebras and Operator Theory*, Academic Press, Boston, 1990.
- [2] F. J. Murray and J. von Neumann, On rings of operators. IV, *Ann. of Math.* **44** (1943), 716-808. DOI 10.2307/1969107.
- [3] A. Connes, Une classification des facteurs de type III, *Ann. Sci. École Norm. Sup.* **6** (1973), 133-252. DOI 10.24033/asens.1247.
- [4] A. Connes, Classification of injective factors. Cases II_1 , II_∞ , III_λ , $\lambda \neq 1$, *Ann. of Math.* **104** (1976), 73-115. DOI 10.2307/1971057.
- [5] U. Haagerup, Connes' bicentralizer problem and uniqueness of the injective factor of type III_1 , *Acta Math.* **158** (1987), 95-148. DOI 10.1007/BF02392257.
- [6] A. S. Toms, On the classification problem for nuclear C^* -algebras, arXiv:math/0509103; *Ann. of Math.* **167** (2008), 1029-1044.

Table 5: Code evidence for the computational claims, with the file and function that demonstrate each claim.

Claim	File	Function
Isomorphism criterion matches enumerated classes up to dimension 16	<code>finite_dim_cstar_decomposition.py</code>	<code>isomorphism_criterion</code>
Algebra counts match theorem 3.6	<code>finite_dim_cstar_decomposition.py</code>	<code>classify_by_dimension</code>
Tensor products multiply block sizes	<code>finite_dim_cstar_decomposition.py</code>	<code>tensor_dimension_vector</code>
Kraus rank equals Choi rank	<code>stinespring_kraus_rank.py</code>	<code>kraus_rank</code>
Minimal Kraus presentations reduce correctly	<code>stinespring_kraus_rank.py</code>	<code>minimize_kraus</code>
Equal Kraus rank does not classify channels	<code>stinespring_kraus_rank.py</code>	<code>same_rank_distinct_channels</code>
Petz forms are positive on sampled tangent vectors	<code>petz_monotone_metrics.py</code>	<code>check_monotone_positive</code>
Distinct functions give distinct metrics	<code>petz_monotone_metrics.py</code>	<code>distinct_functions_distinct_metrics</code>
Taxonomy instances cover the volume's systems	<code>Instances.hs</code>	instance declarations

- [7] D. McDuff, Uncountably many II_1 factors, *Ann. of Math.* **90** (1969), 372-377. DOI 10.2307/1970730.
- [8] W. F. Stinespring, Positive functions on C^* -algebras, *Proc. Amer. Math. Soc.* **6** (1955), 211-216. DOI 10.1090/S0002-9939-1955-0069403-4.
- [9] M.-D. Choi, Completely positive linear maps on complex matrices, *Linear Algebra Appl.* **10** (1975), 285-290. DOI 10.1016/0024-3795(75)90075-0.
- [10] K. Kraus, *States, Effects, and Operations: Fundamental Notions of Quantum Theory*, Lecture Notes in Physics, vol. 190, Springer, Berlin, 1983. DOI 10.1007/3-540-12732-1.
- [11] S. Doplicher, R. Haag, and J. E. Roberts, Fields, observables and gauge transformations. I, *Comm. Math. Phys.* **13** (1969), 1-23. DOI 10.1007/BF01645267.
- [12] S. Doplicher, R. Haag, and J. E. Roberts, Fields, observables and gauge transformations. II, *Comm. Math. Phys.* **15** (1969), 173-200. DOI 10.1007/BF01645674.
- [13] S. Doplicher and J. E. Roberts, A new duality theory for compact groups, *Invent. Math.* **98** (1989), 157-218. DOI 10.1007/BF01388849.
- [14] S. Doplicher and J. E. Roberts, Why there is a field algebra with a compact gauge group describing the superselection structure in particle physics, *Comm. Math. Phys.* **131** (1990), 51-107. DOI 10.1007/BF02097680.
- [15] P. Deligne, Catégories tensorielles, *Mosc. Math. J.* **2** (2002), 227-248. DOI 10.17323/1609-4514-2002-2-2-227-248.
- [16] P. Etingof and S. Gelaki, Isocategorical groups, arXiv:math/0007196.

- [17] V. Ostrik, Module categories, weak Hopf algebras and modular invariants, arXiv:math/0111139; *Transform. Groups* **8** (2003), 177-206. DOI 10.1007/s00031-003-0515-6.
- [18] P. Etingof, D. Nikshych, and V. Ostrik, On fusion categories, arXiv:math/0203060; *Ann. of Math.* **162** (2005), no. 2, 581-642. DOI 10.4007/annals.2005.162.581.
- [19] D. Petz, Monotone metrics on matrix spaces, *Linear Algebra Appl.* **244** (1996), 81-96. DOI 10.1016/0024-3795(94)00211-8.
- [20] D. Petz and Cs. Sudár, Extending the Fisher metric to density matrices, in *Geometry in Present Days Science* (O. E. Barndorff-Nielsen and E. B. Vendel Jensen, eds.), World Scientific, 1999, pp. 21-34; arXiv:quant-ph/0102132.