

Classification of Classical Physical Systems

Part 5 of *The Classification of Mathematical Representations of Physical Systems*

Matthew Long

The YonedaAI Collaboration, YonedaAI Research Collective

Chicago, Illinois, USA

m.long@yonedalabs.ai

<https://yonedalabs.ai>

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Abstract

This paper classifies restricted classes of classical physical systems: symplectic and Poisson systems and classical statistical systems. The central structural fact is an inversion of the Riemannian situation. Darboux's theorem removes every local symplectic invariant, so the classification content of Hamiltonian mechanics is global and topological. On the Poisson side, Weinstein's splitting theorem replaces local rigidity by a rank stratification into symplectic leaves and transverse data, and coadjoint orbits tie the homogeneous case to the Dynkin classification of simple Lie algebras and the highest-weight classification of their finite-dimensional representations. On the statistical side, Chentsov's theorem singles out the Fisher metric as the unique metric monotone under stochastic maps, while the quantum analogue is parametric rather than unique. The results include one complete local classification theorem, one rank normal form, one uniqueness theorem, and two restricted global classification theorems, each stated with hypotheses and proof or proof reference. The limitation that matters most is scope: no global classification theorem for closed symplectic manifolds is claimed or available, decomposition uniqueness is open outside restricted subclasses, and the deformation-quantization bridge to quantum theory remains conjectural. Computational experiments exhibit the local normal forms and supply executable models; they are evidence, not proofs.

1 Introduction

Classical physical systems carry their state, observables, dynamics, symmetries, and composition rules on smooth manifolds and probability simplices, and the relevant mathematical structures are fully developed: symplectic manifolds, Poisson manifolds, probability simplices. A classification of these structures is a question about what the surviving invariants are.

The answer is shaped by a single theorem. Darboux's theorem states that all symplectic manifolds of a fixed dimension are locally indistinguishable: near every point, a symplectic manifold looks exactly like a standard linear space with a standard two-form. Riemannian geometry classifies locally through curvature, an invariant with content at every point. Symplectic geometry has no local invariants at all. Whatever classification content Hamiltonian mechanics possesses lives in the large, in topology and global geometry. This inversion, recorded as a structural lesson of the research program, organizes the entire paper.

Three bodies of established mathematics carry the classification effort. The local theory of Poisson manifolds contributes the rank stratification: Weinstein's splitting theorem decomposes a

neighborhood of a point into a symplectic factor and a transverse Poisson factor, so Poisson geometry replaces the missing local invariants of the symplectic case by a rank stratification and a foliation into symplectic leaves. The theory of coadjoint orbits contributes the homogeneous case. Coadjoint orbits of a Lie group carry a canonical symplectic form, the Kirillov-Kostant-Souriau form, and for compact semisimple groups the orbits form flag manifolds parameterized by the positive Weyl chamber, with dominant integral weights selecting the integral orbits. The Killing-Cartan classification of simple Lie algebras and the highest-weight classification of their representations therefore enter classical mechanics as the classification of homogeneous symplectic manifolds. Information geometry on the classical side contributes the uniqueness theorem: Chentsov's theorem singles out the Fisher metric, up to a constant, as the unique Riemannian metric monotone under stochastic maps. Classical statistical systems thus possess a uniqueness theorem that their quantum analogues provably lack.

No global classification theorem for closed symplectic manifolds is known or claimed. Decomposition of classical systems into irreducible factors is unique only in restricted subclasses, and action-angle monodromy is an explicit obstruction to global decomposition in one of those subclasses. The deformation-quantization bridge between classical and quantum families is a programmatic conjecture, and the paper treats it as conditional throughout. Every theorem below is stated with its hypotheses; every computational claim is marked as evidence rather than proof.

Mathematical equivalence for these systems is structure-preserving isomorphism: symplectomorphisms preserve symplectic forms, Poisson isomorphisms preserve brackets, and reparameterizations preserve probability structure. Physical equivalence is a coarser relation that this paper does not define, and every classification theorem below applies to mathematical structures under these isomorphisms.

2 Mathematical framework

A classical system in this paper is an instance of the five-component schema of the series: a tuple of state space, observables, dynamics, symmetries, and composition rules, with each component specialized to smooth-geometric or measure-theoretic data.

Definition 2.1 (Classical system). A classical system is a tuple $P = (X, O, D, S, C)$ where X is a smooth state space, O is a distinguished family of observables on X , D is a family of admissible state evolutions with their composition rules (vector fields with flows on smooth manifolds, stochastic maps on simplices), S is symmetry data preserving O and D (a Lie group action in the smooth case, outcome permutations and sufficient-statistics embeddings in the statistical case), and C assigns composition data to products of systems. The tuple is the classical instance of the five-component schema of the series.

Definition 2.2 (Symplectic system). A symplectic system is a classical system whose state space is a symplectic manifold (M, ω) of dimension $2n$, whose observables are $O = C^\infty(M)$, and whose dynamics are Hamiltonian vector fields X_H defined by $\iota_{X_H}\omega = dH$ for Hamiltonians $H \in C^\infty(M)$. The Poisson bracket is $\{f, g\} = \omega(X_f, X_g)$ and the flow equation reads $\dot{f} = \{f, H\}$.

Definition 2.3 (Poisson system). A Poisson system is a classical system whose state space is a Poisson manifold (M, π) : a smooth manifold with a bivector field $\pi \in \Gamma(\Lambda^2 TM)$ whose Schouten-Nijenhuis bracket satisfies $[\pi, \pi] = 0$. The bracket $\{f, g\} = \pi(df, dg)$ makes $C^\infty(M)$ a Lie algebra obeying the Leibniz rule in each argument. Symplectic systems are the special case in which π is nondegenerate.

Definition 2.4 (Classical statistical system). A classical statistical system is a classical system whose state space is the interior $\Delta^\circ = \{p \in \mathbb{R}^n : p_i > 0, \sum_i p_i = 1\}$ of the probability simplex

Symbol	Meaning
(M, ω)	symplectic manifold of dimension $2n$ with closed nondegenerate two-form
$\pi, \pi^\sharp$	Poisson bivector and the map $T^*M \rightarrow TM$ it induces
X_H	Hamiltonian vector field defined by $\iota_{X_H}\omega = dH$
$\{f, g\}$	Poisson bracket of observables
$\mathcal{O}_\xi$	coadjoint orbit through $\xi \in \mathfrak{g}^*$
J	momentum map of a symplectic group action
Δ°	interior of the probability simplex
g_p	Fisher metric at the probability vector p
T^*g_{Tp}	pullback of the metric at Tp along a stochastic map
C	product composition of classical systems

Table 1: Notation used throughout the paper. Each symbol is fixed at its first appearance and reused without change.

$\Delta^{n-1} = \{p \in \mathbb{R}^n : p_i \geq 0, \sum_i p_i = 1\}$, whose observables are random variables (functions on the outcome set), and whose admissible state changes are the column-stochastic matrices T that map Δ° into itself (columns strictly positive); maps that reach the boundary are excluded because the Fisher metric is singular there. Restriction to submodels is by sufficient statistics, and the symmetries S are the outcome permutations together with those embeddings.

The three families in force are the symplectic and Poisson family, the smooth geometric family, and the classical probability and measure family. The quantum side of the probability family appears in Section 8 only as a contrast.

Definition 2.5 (Morphisms and equivalence in force). Between symplectic systems the admissible exact morphisms are symplectomorphisms and, between Poisson systems, Poisson maps (smooth maps pulling Poisson brackets back). Between classical statistical systems the admissible coarse-grainings are stochastic maps. Mathematical equivalence of classical systems is isomorphism in the ambient category fixed here: symplectomorphism, Poisson isomorphism, or diffeomorphism of statistical models carrying the model map. Every such identification preserves the symplectic form or the Poisson bracket or the probability structure, and the paper names the preserved layer whenever an identification is claimed.

Definition 2.6 (Irreducibility relative to composition). Fix the product composition of Section 9 as the composition data C . A classical system is C -irreducible if it admits no decomposition $P \simeq P_1 \times P_2$ with both factors of positive dimension. Irreducibility is always relative to the composition data chosen, and the paper never asserts irreducibility without naming that data.

Two limitations of the framework are load-bearing. First, the admissibility conditions of the series are not yet fixed, so every statement below quantifies over explicitly listed classes rather than over all admissible systems. Second, mathematical equivalence is strictly finer than physical equivalence, and no physical equivalence for classical systems is fixed here. The standing qualification of the series governs any attempt to derive physical conclusions from relational data: “Yoneda does not, by itself, establish physical equivalence, empirical adequacy, or a classification theorem.”

Definition 2.7 (Canonical transformation and coarse-graining). A canonical transformation between symplectic systems (M_1, ω_1) and (M_2, ω_2) is a diffeomorphism $\varphi : M_1 \rightarrow M_2$ with $\varphi^*\omega_2 = \omega_1$; it carries Hamiltonian flows to Hamiltonian flows and preserves the Poisson bracket. A coarse-graining

of a classical statistical system is a stochastic map T ; it preserves the probability structure and contracts the Fisher metric by theorem 8.3. The two kinds of morphism implement the two equivalence regimes of this paper: exact mathematical equivalence on the symplectic side and a metric-degrading order relation on the statistical side.

3 Local classification of symplectic systems

The classification of symplectic manifolds is locally trivial and globally hard. The local triviality is a theorem.

Lemma 3.1 (Linearization at a point). *Let (M^{2n}, ω) be symplectic and let $p \in M$. There are coordinates $(x_1, y_1, \dots, x_n, y_n)$ centered at p in which $\omega_p = \sum_{i=1}^n dx_i \wedge dy_i$ at p .*

Proof. Choose any coordinates $u_1, \dots, u_{2n}$ centered at p . The matrix $\Omega = (\omega_{ij}(p))$ of ω at p is skew-symmetric and invertible. Symplectic Gram-Schmidt: pick u_1 with $\omega(u_1, \cdot) \neq 0$, set v_1 with $\omega(u_1, v_1) = 1$, restrict to the symplectic complement of $\text{span}(u_1, v_1)$, and iterate. The resulting frame $e_1, f_1, \dots, e_n, f_n$ satisfies $\omega(e_i, f_j) = \delta_{ij}$, $\omega(e_i, e_j) = \omega(f_i, f_j) = 0$, and dual coordinates give the claim. $\square$

Lemma 3.2 (Moser trick on a ball). *Let B be an open ball centered at p in a chart, and let ω_t , $t \in [0, 1]$, be a smooth family of symplectic forms on B with ω_t nondegenerate for all t . If $\dot{\omega}_t$ is exact for each t , then there is a concentric ball B' compactly contained in B carrying an isotopy φ_t with $\varphi_t^* \omega_t = \omega_0$ for all t , and φ_1 maps B' into B .*

Proof. Let Z be the radial vector field of B from its center p , so $Z(p) = 0$ and B is star-shaped for Z . For the closed form $\dot{\omega}_t$ the homotopy operator gives the primitive $\alpha_t = \iota_Z \dot{\omega}_t$, which satisfies $d\alpha_t = \dot{\omega}_t$ and $\alpha_t(p) = 0$ because Z vanishes at p . Solve $\iota_{X_t} \omega_t = -\alpha_t$; nondegeneracy of ω_t makes this solvable and $X_t(p) = 0$. Let L be a Lipschitz constant for the field X_t on B , uniform in $t \in [0, 1]$, and choose $r > 0$ with $B(p, r)$ compactly contained in B and $e^L r$ smaller than the distance from p to the boundary of B . Grönwall's inequality gives $|\varphi_t(x) - p| \leq e^{Lt}|x - p|$ for as long as the trajectory stays in B , so the flow φ_t of X_t starting at the identity exists on all of $[0, 1]$ and maps $B' = B(p, r)$ into B . Then

$$\frac{d}{dt} \varphi_t^* \omega_t = \varphi_t^* (\dot{\omega}_t + \mathcal{L}_{X_t} \omega_t) = \varphi_t^* (d\alpha_t + d\iota_{X_t} \omega_t) = \varphi_t^* (d\alpha_t - d\alpha_t) = 0,$$

so $\varphi_t^* \omega_t$ is constant and equals ω_0 . $\square$

Theorem 3.3 (Darboux). *Let (M^{2n}, ω) be a symplectic manifold and $p \in M$. There exists a neighborhood U of p and a diffeomorphism $\varphi : U \rightarrow V$ onto an open subset $V \subset \mathbb{R}^{2n}$ such that $\varphi^* \omega_0 = \omega|_U$ where $\omega_0 = \sum_{i=1}^n dx_i \wedge dy_i$. In particular, any two symplectic manifolds of the same dimension are locally symplectomorphic.*

Proof. By Theorem 3.1 choose centered coordinates on a ball B around p in which $\omega_p = \omega_0$ at the center. The family $\omega_t = \omega_0 + t(\omega - \omega_0)$ is a smooth family of closed two-forms with ω_t nondegenerate on a possibly smaller ball $B' \subset B$ (nondegeneracy is an open condition and holds at p for every t). On B' the form $\omega - \omega_0$ is closed, and since B' is contractible its de Rham cohomology vanishes in degree two, so $\omega - \omega_0 = d\alpha$ for a one-form α and $\dot{\omega}_t = d\alpha$ is exact for every t . Theorem 3.2 provides an isotopy φ_t of B' with $\varphi_1^* \omega_1 = \omega_0$ and $\omega_1 = \omega$. The diffeomorphism φ_1^{-1} exhibits (B', ω) as a symplectomorphic image of an open set in $(\mathbb{R}^{2n}, \omega_0)$. $\square$

Corollary 3.4 (Absence of local invariants). *Every local symplectic invariant of a $2n$ -dimensional symplectic manifold is constant on connected components. Any classification of symplectic manifolds up to symplectomorphism must be carried by global data.*

Proof. Local symplectic invariants are preserved by symplectomorphisms, and theorem 3.3 supplies a symplectomorphism from a neighborhood of any point onto a neighborhood of any point of the standard space of the same dimension. Two points of a connected manifold are joined by a chain of such neighborhoods. $\square$

Remark 3.5 (The inversion). In Riemannian geometry the curvature tensor is a local invariant with pointwise content, and local normal forms are scarce. In symplectic geometry the local classification has one entry per dimension, and the classification problem moves to global topology.

The same argument classifies exact symplectic structures locally in the strongest sense: on a ball, every symplectic form is not merely standardizable but symplectomorphic to the constant standard form.

4 Poisson geometry and local models

Poisson manifolds retain local structure where symplectic manifolds have none. The rank of π at a point is the invariant Darboux's theorem destroyed for the symplectic case, and a normal form organizes a neighborhood of a point around that rank.

Theorem 4.1 (Weinstein splitting). *Let (M, π) be a Poisson manifold and $p \in M$ a point of rank $2k$. There exist a neighborhood U of p and a Poisson diffeomorphism from U onto a product $U_1 \times U_2$ where U_1 is a neighborhood of a point in a symplectic manifold (N^{2k}, ω) and U_2 is a neighborhood of a zero q_0 of a Poisson structure π_2 of rank zero at q_0 .*

Proof sketch. The symplectic leaf S through p is an immersed submanifold of dimension $2k$ and carries the induced symplectic form ω_S characterized by $\omega_S^b = (\pi|_S)^{-1}$. In coordinates adapted to the leaf the bivector takes the form $\pi = \pi_S + \pi_T$ with π_S nondegenerate along S and π_T tangent to the transverse fibers and vanishing on the zero section. The transverse factor π_T retains its higher-order terms at the zero q_0 ; the splitting theorem does not linearize it, and smooth linearization of transverse Poisson structures fails in general. The full argument appears in Weinstein [14]; the product form stated here is the part used below. $\square$

Proposition 4.2 (Leaf foliation). *A Poisson manifold (M, π) is foliated by its symplectic leaves: maximal immersed submanifolds $S \subset M$ on which $\pi|_S$ is nondegenerate and which are tangent to the distribution $\text{im } \pi^\sharp$. Each leaf is a symplectic manifold, and the restriction of the Poisson bracket of functions to a leaf is the bracket of the leaf symplectic form.*

Proof. The distribution $x \mapsto \text{im } \pi_x^\sharp$ is integrable in the sense of Stefan-Sussmann precisely because $[\pi, \pi] = 0$: the vanishing of the Schouten-Nijenhuis bracket is the integrability condition for the associated Lie algebroid, and its leaves are the symplectic leaves. On a leaf S the map $\pi^\sharp|_S : T^*S \rightarrow TS$ is an isomorphism by maximality of the leaf, and its inverse is ω_S^b ; the bracket identity $\{f, g\}|_S = \omega_S(X_f, X_g)$ is then the definition of ω_S . $\square$

Example 4.3 (Lie-Poisson bracket on $\mathfrak{g}^*$). For a finite-dimensional Lie algebra $\mathfrak{g}$ the dual space $\mathfrak{g}^*$ carries the linear Poisson structure $\pi(\xi)(df, dg) = \langle \xi, [df(\xi), dg(\xi)] \rangle$ under the identification $T_\xi^* \mathfrak{g}^* \cong \mathfrak{g}$. Its symplectic leaves are the coadjoint orbits. For $\mathfrak{g} = \mathfrak{so}(3)$ the space $\mathfrak{g}^* \cong \mathbb{R}^3$ carries the angular momentum bracket $\{x, y\} = z$ and its cyclic permutations; the rank is two away from the origin and

zero at the origin, the leaves are the spheres $\{|\xi| = r\}$ together with the origin, and theorem 4.1 at a point of a sphere recovers a two-dimensional symplectic factor with a one-dimensional transverse Poisson factor.

Theorem 4.4 (Kirillov-Kostant-Souriau form). *Let G be a Lie group with Lie algebra $\mathfrak{g}$ and let $\mathcal{O} \subset \mathfrak{g}^*$ be a coadjoint orbit. The formula*

$$\omega_\xi(\text{ad}_x^* \xi, \text{ad}_y^* \xi) = \langle \xi, [x, y] \rangle$$

defines a G -invariant symplectic form ω_ξ on $\mathcal{O}$, and the coadjoint action of G on $(\mathcal{O}, \omega_\xi)$ is by symplectomorphisms. The Poisson structure induced on $\mathfrak{g}^$ by the Lie bracket restricts on each orbit to the inverse of ω_ξ .*

Proof. The stabilizer $\mathfrak{g}_\xi$ of ξ under the coadjoint action is the kernel of the skew form $(x, y) \mapsto \langle \xi, [x, y] \rangle$, so the form descends to $T_\xi \mathcal{O} \cong \mathfrak{g}/\mathfrak{g}_\xi$. Nondegeneracy on the orbit follows from this identification, and the identity $T_\xi \mathcal{O} = \text{im ad}^*(\xi)$ makes the descended form nondegenerate. Invariance is the identity $\langle \text{Ad}_g^* \xi, [\text{Ad}_g x, \text{Ad}_g y] \rangle = \langle \xi, [x, y] \rangle$ under the adjoint action. The relation between the orbit form and the linear Poisson structure is a direct computation in the Lie-Poisson bracket of theorem 4.3; see Kirillov [9] for the full statement. $\square$

Proposition 4.5 (Casimir functions). *A smooth function $F \in C^\infty(M)$ on a Poisson manifold (M, π) is constant on the symplectic leaves if and only if $\{F, f\} = 0$ for all $f \in C^\infty(M)$. On the Lie-Poisson space $\mathfrak{g}^*$ these functions are exactly the coadjoint-invariant functions $F(\xi) = F(\text{Ad}_g^* \xi)$ for all $g \in G$. For $\mathfrak{g} = \mathfrak{so}(3)$ the Casimir is $F(\xi) = |\xi|^2$, and its level sets are the symplectic leaves of theorem 4.3.*

Proof. The bracket $\{F, f\} = \pi(dF, df)$ vanishes for all f if and only if dF annihilates the image of $\pi^\sharp$, and the image of $\pi^\sharp$ spans the tangent spaces of the leaves. So F has vanishing differential along the leaves, which on each connected leaf means F is constant. On $\mathfrak{g}^*$ the condition $\{F, f\} = 0$ for all linear functions $f(\eta) = \langle \eta, x \rangle$ reads $\langle \xi, [\nabla F(\xi), x] \rangle = 0$ for all x , which is the infinitesimal form of coadjoint invariance. For $\mathfrak{so}(3)$ the invariant polynomials are generated by the quadratic form $|\xi|^2$, and theorem 4.3 identifies its level sets with the leaves. $\square$

Casimir functions are the first integral data of Poisson dynamics: a Hamiltonian that is itself a Casimir generates no motion, and a function in involution with the Casimirs descends to a function on the leaf space.

Theorem 4.4 is the bridge from classical mechanics to the classification theory of Lie algebras. The symplectic leaves of the Lie-Poisson structure on $\mathfrak{g}^*$ are the coadjoint orbits, the orbits of a compact semisimple group are classified by combinatorial data of the Dynkin diagram, and Section 6 makes that classification precise.

5 Global invariants of symplectic systems

The classification content that survives Darboux's theorem is global.

Theorem 5.1 (Moser stability). *Let M be a closed manifold and ω_t , $t \in [0, 1]$, a smooth family of symplectic forms with $[\omega_t] \in H^2(M; \mathbb{R})$ independent of t . Then there is an isotopy φ_t of M with $\varphi_t^* \omega_t = \omega_0$ for all t .*

Proof. Constancy of the cohomology class gives $\dot{\omega}_t = d\alpha_t$ for a smooth family of one-forms α_t . Solve $\iota_{X_t}\omega_t = -\alpha_t$ and integrate X_t to an isotopy φ_t of the closed manifold M . The computation of theorem 3.2 gives $\frac{d}{dt}\varphi_t^*\omega_t = 0$ without any support condition, so $\varphi_1^*\omega_1 = \omega_0$. The argument is due to Moser [11]. $\square$

Corollary 5.2 (Classification refines diffeomorphism and cohomology). *Let $f : M \rightarrow M'$ be a diffeomorphism of closed symplectic manifolds carrying $[\omega]$ to $[\omega']$. If there is a smooth path of symplectic forms ω_t in the constant class $[\omega]$ connecting ω to $f^*\omega'$, then (M, ω) and (M', ω') are symplectomorphic. Symplectomorphism classes therefore form a strictly finer partition than the pairs consisting of a diffeomorphism type and an orbit of $[\omega]$ under the diffeomorphism group.*

The cohomology class $[\omega] \in H^2(M; \mathbb{R})$ is therefore a classifying invariant in the restricted sense of the series, together with the diffeomorphism invariants of M and the action of $\text{Diff}(M)$ on $H^2(M; \mathbb{R})$. For two-dimensional symplectic manifolds this is a complete classification.

Example 5.3 (Surfaces). A symplectic form on a closed surface is an area form, and the total area $\int_M \omega$ is a symplectomorphism invariant. Two closed oriented surfaces with area forms are symplectomorphic if and only if they are diffeomorphic and the areas agree. Classification here is classification of genus and area.

Example 5.4 (Area forms on the sphere). Let σ and σ' be two area forms on S^2 with the same total area $\int_{S^2} \sigma = \int_{S^2} \sigma'$. Then (S^2, σ) and (S^2, σ') are symplectomorphic. The interpolation $\sigma_t = (1-t)\sigma + t\sigma'$ consists of area forms of the same total area, hence symplectic forms with constant cohomology class in $H^2(S^2; \mathbb{R})$, and theorem 5.1 above produces the isotopy. The classification of area forms on a fixed surface up to symplectomorphism is therefore the classification of total areas, in agreement with theorem 5.3, and the theorem in action is Moser stability rather than any local analysis.

Example 5.5 (Products of spheres). On $M = S^2 \times S^2$ with area forms σ_1 and σ_2 of total area one on each factor, consider $\omega_{a,b} = a\sigma_1 \oplus b\sigma_2$ with $a, b > 0$. The cohomology class $[\omega_{a,b}] = a[\sigma_1] + b[\sigma_2]$ is a symplectomorphism invariant, and the diffeomorphism group of $S^2 \times S^2$ acts on $H^2(M; \mathbb{R})$ by permuting the two factors. Hence $(M, \omega_{a,b})$ is symplectomorphic to $(M, \omega_{a',b'})$ if and only if $\{a, b\} = \{a', b'\}$ as unordered pairs. The classification is the unordered pair of areas, a two-dimensional global invariant with no local content.

Beyond the cohomology class, the standard global invariants are the characteristic classes of M : any symplectic manifold admits an ω -compatible almost complex structure J , and the Chern classes $c_i(M, J)$ are independent of the choice of J . For closed four-manifolds these classes interact with deep rigidity phenomena, and no complete classification is claimed in this paper.

In dimension four, global symplectic classification refines smooth four-manifold topology and remains incomplete outside toric and product regimes. De Rham cohomology distinguishes the standard symplectic structures on $\mathbb{CP}^2$ with its Fubini-Study form, T^4 with a constant form, and the products of theorem 5.5, and theorem 9.8 classifies the toric four-manifolds, including the one-point blowup of $\mathbb{CP}^2$, by polytopes. For general closed four-manifolds, symplectomorphism classification is finer than the diffeomorphism type together with the action of $\text{Diff}(M)$ on $H^2(M; \mathbb{R})$, and distinguishing smooth structures that share these classical invariants uses gauge-theoretic and pseudoholomorphic invariants (Seiberg-Witten and Taubes-type invariants) that this paper does not develop.

The strongest global rigidity in the symplectic category is metric-free.

Definition 5.6 (Symplectic capacity). A symplectic capacity is a rule c assigning to each symplectic manifold (M^{2n}, ω) a number $c(M, \omega) \in [0, +\infty]$ such that: (i) monotonicity holds, $c(M, \omega) \leq$

$c(M', \omega')$ whenever a symplectomorphism embeds (M, ω) into (M', ω') ; (ii) conformality holds, $c(M, \lambda\omega) = |\lambda|c(M, \omega)$ for $\lambda \neq 0$; (iii) normalization holds, $c(B^{2n}(1)) = c(B^2(1) \times \mathbb{R}^{2n-2}) = \pi$ where $B^{2n}(1)$ is the open unit ball and $B^2(1)$ the open unit disc.

Proposition 5.7 (Gromov width). *The Gromov width*

$$w_G(M, \omega) = \sup\{\pi r^2 : B^{2n}(r) \text{ embeds symplectically into } (M, \omega)\}$$

is a symplectic capacity in the sense of theorem 5.6.

Proof reference. Monotonicity is immediate from composition of embeddings, and conformality from the scaling of the standard form. The normalization and the capacity axiom for disjoint unions of balls follow from the embedding theory of Gromov [6]; the width is the first capacity constructed. $\square$

Gromov's nonsqueezing theorem is the rigidity statement that makes capacities genuine invariants rather than convenient notation.

Theorem 5.8 (Gromov nonsqueezing). *Let $B^{2n}(r)$ be the open ball of radius r in $\mathbb{R}^{2n}$ and $B^2(R) \times \mathbb{R}^{2n-2}$ the cylinder of radius R . A symplectic embedding $B^{2n}(r) \hookrightarrow B^2(R) \times \mathbb{R}^{2n-2}$ exists only if $r \leq R$.*

Proof reference. The proof uses J -holomorphic curves for a suitable compatible almost complex structure and is due to Gromov [6]. The conclusion is the source of genuine global symplectic rigidity and of the capacity invariants. $\square$

Nonsqueezing has no Riemannian analogue: a round ball embeds into a thin cylinder in ordinary Euclidean geometry at every scale. The invariant content of symplectic geometry is therefore both global and metric-free.

On open manifolds the situation reverses. By the h -principle for symplectic structures on open manifolds, every open manifold admits an exact symplectic form, and open symplectic manifolds are classified up to symplectomorphism by diffeomorphism type [7]. Closed manifolds are rigid; open manifolds are flexible. Any classification theorem for classical systems must state which regime it addresses.

6 Coadjoint orbits and homogeneous symplectic manifolds

The homogeneous symplectic manifolds of classical mechanics are the coadjoint orbits of Theorem 4.4, and their classification inherits the combinatorics of the Killing-Cartan theory of simple Lie algebras and the highest-weight theory of their representations.

Definition 6.1 (Momentum map and homogeneous orbits). Let G act symplectically on (M, ω) with momentum map $J : M \rightarrow \mathfrak{g}^*$ characterized by $\langle dJ, x \rangle = \iota_{x_M} \omega$ for the fundamental vector fields x_M . The fibers and orbits of J are the symmetry-level objects of the classical system, and coadjoint orbits in $\mathfrak{g}^*$ are the model homogeneous symplectic manifolds.

Theorem 6.2 (Coadjoint orbits of compact semisimple groups). *Let G be compact, connected, and semisimple with complexification $G^{\mathbb{C}}$ and maximal torus T with Lie algebra $\mathfrak{t}$. Every coadjoint orbit of G meets $\mathfrak{t}^*$ in a single W -orbit, so the orbits are parameterized continuously by points ξ of the closed positive Weyl chamber $\mathfrak{t}_+^*$. Each orbit is a flag manifold $G^{\mathbb{C}}/P_{\xi}$ where the parabolic P_{ξ} is determined by the support of ξ , that is by the subset of Dynkin nodes on whose walls ξ lies; regular ξ gives full flag manifolds and singular ξ gives partial flags. The symplectic form on the orbit is the*

Diagram	Complex simple Lie algebra	Dimension
A_n	$\mathfrak{sl}(n+1, \mathbb{C})$	$n(n+2)$
B_n	$\mathfrak{so}(2n+1, \mathbb{C})$	$n(2n+1)$
C_n	$\mathfrak{sp}(2n, \mathbb{C})$	$n(2n+1)$
D_n	$\mathfrak{so}(2n, \mathbb{C})$	$n(2n-1)$
G_2	exceptional algebra of type G_2	14
F_4	exceptional algebra of type F_4	52
E_6	exceptional algebra of type E_6	78
E_7	exceptional algebra of type E_7	133
E_8	exceptional algebra of type E_8	248

Table 2: The Killing-Cartan list of complex simple Lie algebras. The classification of homogeneous symplectic manifolds of theorem 6.2 inherits this list and adds, per diagram, the positive Weyl chamber of continuous orbit parameters and the dominant integral weights selecting the integral orbits.

Kirillov-Kostant-Souriau form of Theorem 4.4. The dominant integral weights form the discrete sublattice $\Lambda_+ \subset \mathfrak{t}_+^*$ that selects the integral, that is quantizable, orbits; those are the orbits carrying finite-dimensional irreducible representations by the Borel-Weil theorem.

Proof reference. The Killing-Cartan classification of complex simple Lie algebras is the list of Dynkin diagrams $A_n, B_n, C_n, D_n, G_2, F_4, E_6, E_7, E_8$ [8]. The parameterization of coadjoint orbits by the positive Weyl chamber is the standard maximal-torus argument, and the identification of orbits with $G^\mathbb{C}/P_\xi$ is the complex-geometric form of the same statement [9]. The highest-weight theorem classifies the finite-dimensional irreducible representations of $\mathfrak{g}^\mathbb{C}$ by dominant integral weights [8], and the Borel-Weil theorem realizes each such representation on the corresponding integral orbit. The paper uses these results as established background and reproduces none of the classification lists. $\square$

Example 6.3 (The spheres as coadjoint orbits). For $G = SU(2)$ the dual $\mathfrak{su}(2)^* \cong \mathbb{R}^3$, and the coadjoint orbits are the spheres $\{|\xi| = r\}$ of every radius $r > 0$ together with the degenerate orbit at the origin: a continuous one-parameter family, the Weyl chamber being the half-line $r \geq 0$. The Kirillov-Kostant-Souriau form of Theorem 4.4 on the sphere of radius r is $1/r$ times the round area form, of total symplectic area $4\pi r$. The integral orbits are those with $2r$ a nonnegative integer, and the Borel-Weil theorem attaches to the orbit of radius $r = j$ the spin- j representation of dimension $2j+1$. The example separates the two classification statements: the classical orbit family is continuous in r , and the representation labels form the discrete sublattice $r \in \frac{1}{2}\mathbb{Z}_{\geq 0}$.

The classification content of theorem 6.2 is finite-dimensional and combinatorial: Dynkin diagram nodes and dominant weights. The theorem classifies homogeneous symplectic manifolds under a group with transitive action, not arbitrary symplectic manifolds. The orbit method program, which seeks to extend this dictionary to general representations of general Lie groups through their coadjoint orbits, is not established at the level of an equivalence and is not used as a dependency here.

Table 2 is the classification content that flows into classical mechanics: for each simple algebra the coadjoint orbits form a family parameterized by the positive Weyl chamber of theorem 6.2, the dominant integral weights select the integral sublattice of that chamber, and the symplectic geometry of each orbit is fixed by Theorem 4.4. The exceptional diagrams show that the classification is not a uniform infinite family alone, and the dimensions record that the homogeneous symplectic manifolds attached to exceptional algebras are concrete finite-dimensional objects.

7 Hamiltonian dynamics and integrable systems

The dynamical layer of a classical system is the family of Hamiltonian flows. The classification of dynamics is far from complete, but the integrable case admits normal forms with explicit invariants and explicit obstructions to global classification.

Definition 7.1 (Integrable system). A Liouville integrable system is a symplectic manifold (M^{2n}, ω) with n functions $H_1, \dots, H_n \in C^\infty(M)$ in involution, $\{H_i, H_j\} = 0$, and with $dH_1 \wedge \dots \wedge dH_n$ nonzero on a dense open subset of M .

Theorem 7.2 (Liouville-Arnold). Let $(M^{2n}, \omega, H_1, \dots, H_n)$ be a Liouville integrable system and let $M_c = \{H_1 = c_1, \dots, H_n = c_n\}$ be a compact connected regular level set. Then M_c is diffeomorphic to an n -torus T^n , and a neighborhood of M_c in M carries action-angle coordinates (I, θ) in which $\omega = \sum_i dI_i \wedge d\theta_i$ and the H_i depend on the actions I_i alone.

Proof reference. The Hamiltonian flows of the H_i commute and preserve M_c , giving a locally free T^n -action whose orbits fill M_c ; compactness and connectedness make the action transitive. The action variables are the periods of ω over a basis of $H_1(M_c; \mathbb{Z})$, and the angle variables are coordinates on the fibers. The full proof appears in Arnold [2]. $\square$

Proposition 7.3 (Action variables are periods). In the action-angle coordinates of theorem 7.2, the action variables are the periods of the action one-form λ with $d\lambda = \omega$ over a basis $\gamma_1, \dots, \gamma_n$ of $H_1(M_c; \mathbb{Z})$:

$$I_j = \frac{1}{2\pi} \oint_{\gamma_j} \lambda.$$

The value depends on the choice of primitive λ : two primitives differ by a closed one-form, and the resulting shift of the actions is the period vector of that form, a vector of $H^1(M_c; \mathbb{R}) \cong \mathbb{R}^n$. Once λ is fixed, the angle variables are fiberwise coordinates normalized so that $\omega = \sum_j dI_j \wedge d\theta_j$.

Proof. On a neighborhood of M_c the form ω is exact, $\omega = d\lambda$, and the n commuting flows of the H_i generate a torus action. The periods of λ over the cycles γ_j are constant on the base of the fibration because the cycles are covariantly constant under the flows, and the identity $I_j = \frac{1}{2\pi} \oint_{\gamma_j} \lambda$ is the normalization that yields $\omega = \sum_j dI_j \wedge d\theta_j$. Replacing λ by $\lambda + \eta$ with η closed shifts each I_j by $\frac{1}{2\pi} \oint_{\gamma_j} \eta$, and those shifts range over all of $\mathbb{R}^n$. $\square$

Action-angle variables give a local decomposition of an integrable system into n one-degree-of-freedom systems. Global action-angle coordinates need not exist: the period lattice of the torus fibration can have monodromy.

Proposition 7.4 (Monodromy obstruction). There exist integrable Hamiltonian systems whose regular compact fibers are tori and which admit no global action-angle coordinates; the obstruction is the monodromy representation of π_1 of the base of the fibration acting on the period lattice $H_1(T^n; \mathbb{Z})$.

Proof reference. Duistermaat [5] computes the monodromy of the period bundle for integrable systems with focus-focus singularities, and the spherical pendulum is the standard example: parallel transport of a fiber around a critical value returns the fiber with its lattice mapped by a nontrivial unipotent matrix. Global action-angle coordinates would trivialize the period bundle, so nontrivial monodromy obstructs them. $\square$

Example 7.5 (Spherical pendulum). The spherical pendulum on T^*S^2 with Hamiltonian $H = \frac{1}{2}(p_\theta^2 + p_\varphi^2 / \sin^2 \theta) - \cos \theta$ is integrable with the vertical angular momentum p_φ as second integral.

Regular compact fibers are two-tori, and the system has one focus-focus critical point. Parallel transport of the period lattice around the critical value acts on $H_1(T^2; \mathbb{Z})$ by the unipotent matrix $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$. The local decomposition of theorem 7.2 holds near every regular fiber, and no global decomposition into action-angle pairs exists on the complement of the critical value.

Example 7.6 (Euler top). The free rigid body is the Hamiltonian system on the coadjoint orbit $\mathcal{O} \subset \mathfrak{so}(3)^*$ of theorem 4.3 with Hamiltonian $H(\xi) = \frac{1}{2}(\xi_1^2/I_1 + \xi_2^2/I_2 + \xi_3^2/I_3)$ for positive moments of inertia I_i . The Casimir $|\xi|^2$ is a second integral in involution with H , so the system is integrable on each sphere $|\xi|^2 = r^2$. The regular compact fibers are the intersections of the energy ellipsoids with the momentum spheres, the trajectories are the Euler-Poinsot curves on those spheres, and theorem 7.2 applies at every regular fiber. The example lives simultaneously in the homogeneous symplectic geometry of Section 6 and in the integrable dynamics of this section: its state space is a coadjoint orbit, its invariant is the Casimir, and its local normal form is action-angle.

Monodromy is the first genuine obstruction in this paper to decomposition: the local decomposition of theorem 7.2 into one-degree-of-freedom factors is a statement about a neighborhood of one fiber, and Theorem 7.4 exhibits integrable systems that do not decompose globally. Stability of integrable dynamics is likewise regime-restricted.

Theorem 7.7 (Nekhoroshev estimate). *Let $H = H_0 + \varepsilon H_1$ be a steep integrable Hamiltonian system with n degrees of freedom and let $H_0(I)$ be the integrable part in action variables. There are positive constants c , a , and b depending on the system such that, for ε small, every trajectory satisfies*

$$\|I(t) - I(0)\| \leq c\varepsilon^b \quad \text{for all } 0 \leq t \leq \exp(c\varepsilon^{-a}).$$

The bound is on the drift of the action variables, not on the energy, which is conserved exactly.

Proof reference. The estimate is due to Nekhoroshev [12]. Steepness is the geometric condition on H_0 under which the normal form of the flow is integrable to high order, and it prevents Arnold diffusion across the resonance planes of action space. The exponential time scale follows from that normal form. The theorem is a statement about the action drift of perturbed integrable systems over exponentially long times. $\square$

8 Classical statistical systems and information geometry

The classical probability family has one uniqueness theorem in the whole inventory of classical classification results. Chentsov's theorem singles out the Fisher metric as the unique metric monotone under stochastic maps, and the quantum analogue of that metric is parametric rather than unique.

Definition 8.1 (Fisher metric). On the open simplex $\Delta^\circ \subset \mathbb{R}^n$ the Fisher metric is the Riemannian metric whose value at p on tangent vectors u, v (with $\sum_i u_i = \sum_i v_i = 0$) is

$$g_p(u, v) = \sum_{i=1}^n \frac{u_i v_i}{p_i}.$$

On a smooth statistical model $\theta \mapsto p_\theta$ the Fisher metric is the pullback of this metric, with components $g_{jk}(\theta) = \sum_i \frac{1}{p_i} \partial_j p_i \partial_k p_i$.

Example 8.2 (Fisher metric on three outcomes). For $p = (\frac{1}{2}, \frac{1}{3}, \frac{1}{6})$ and the tangent vector $u = (1, -1, 0)$ the Fisher metric evaluates to $g_p(u, u) = \frac{1}{1/2} + \frac{1}{1/3} = 5$. At the uniform point $p = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$ the same vector gives $g_p(u, u) = 6$. The metric therefore weights directions by the rarity of the outcomes they move, and the example is the two-dimensional case of theorem 8.1.

Definition 8.3 (Monotone metric). A Riemannian metric g on Δ° is monotone under stochastic maps if for every column-stochastic matrix T and every p , the pullback T^*g_{Tp} at p satisfies $T^*g_{Tp} \leq g_p$ as quadratic forms. Monotonicity is the requirement that coarse-graining cannot increase distinguishability.

Theorem 8.4 (Chentsov). *Let g be a family of Riemannian metrics $g^{(n)}$ on the open simplices $\Delta^\circ \subset \mathbb{R}^n$ for all $n \geq 1$, compatible with Markov embeddings (pullback along every sufficient statistic is an isometry) and monotone under stochastic maps in the sense of theorem 8.3. Then one constant $c > 0$ satisfies $g^{(n)} = c g_{\text{Fisher}}^{(n)}$ for every n , where $g_{\text{Fisher}}^{(n)}$ is the Fisher metric of theorem 8.1. The cross-dimension compatibility is a hypothesis: monotonicity on a single fixed simplex does not determine the metric up to scale.*

Proof structure. Monotonicity forces equality along sufficient statistics, since a sufficient statistic admits a stochastic retraction and monotonicity in both directions along the retraction. Invariance under the sufficient statistics of the simplex reduces the metric to a family $g_p(u, v) = \sum_i \frac{u_i v_i}{p_i} f(p)$ compatible with all embeddings of lower-dimensional simplices, and the compatibility across dimensions forces f to be constant. The complete proof is in Chentsov [3]; the structure stated here is the part that fixes the metric up to scale. $\square$

Lemma 8.5 (Equality along sufficient statistics). *Let g be a monotone metric on Δ° and let T be a stochastic map with a stochastic left inverse S , that is $ST = \text{Id}$. Then T is an isometric embedding for g , and for every tangent vector u at p ,*

$$g_p(u, u) = g_{Tp}(Tu, Tu).$$

Proof. Monotonicity of g for T gives $g_p(u, u) \geq g_{Tp}(Tu, Tu)$, and monotonicity for S gives

$$g_{Tp}(Tu, Tu) \geq g_p(STu, STu) = g_p(u, u).$$

The two inequalities sandwich, so equality holds and T preserves g exactly. This is the equality case that theorem 8.4 uses to reduce the classification to the invariant metrics under the sufficient statistics of the simplex. $\square$

Example 8.6 (Isometries of the Fisher metric). Under the square-root map $x_i = \sqrt{p_i}$ the open simplex Δ° maps to the open positive orthant of the unit sphere, and the Fisher metric becomes four times the round metric of that sphere: $dx_i = u_i/(2\sqrt{p_i})$ gives $\sum_i dx_i^2 = \frac{1}{4}g_p(u, u)$. The isometries of the round sphere preserving the positive orthant are the coordinate permutations, so the isometries of the Fisher metric are exactly the relabelings of the outcome set. Stochastic maps that are not invertible on the model are contractions rather than isometries (theorem 8.3), and theorem 8.5 identifies the case of equality. The example shows concretely how the preserved layers of theorem 2.7 separate exact equivalence from coarse-graining.

Corollary 8.7 (Fisher metric as classifying invariant). *The Fisher metric, up to positive scale, is a classifying invariant of the classical statistical family under stochastic morphisms: it is the unique metric compatible with the coarse-graining order.*

Remark 8.8 (Connections beside the metric). The Fisher metric is the unique monotone Riemannian object, but the statistical manifold carries more structure than its metric. The α -connections of Amari [1] are a one-parameter family of affine connections dual to each other, α and $-\alpha$, with $\alpha = 0$ the Levi-Civita connection of the Fisher metric, and the exponential and mixture families are the $\alpha = 1$ and $\alpha = -1$ flat coordinates. The uniqueness theorem of Chentsov governs the metric alone; the connection family is parametric, and no statement of this paper transfers metric uniqueness to the connections.

Proposition 8.9 (Product of statistical models). *Let $p \in \Delta^\circ \subset \mathbb{R}^n$ and $q \in \Delta^\circ \subset \mathbb{R}^m$ be probability vectors and let $p \otimes q \in \Delta^\circ \subset \mathbb{R}^{nm}$ be their product. Then the Fisher metric of the product is the direct sum of the factors:*

$$g_{p \otimes q}(u \otimes q + p \otimes v, u \otimes q + p \otimes v) = g_p(u, u) + g_q(v, v)$$

for tangent vectors u at p and v at q .

Proof. Write the tangent vector at $p \otimes q$ in the product chart. The probability of the pair (i, j) is $p_i q_j$, and the displacement in that coordinate is $u_i q_j + p_i v_j$. The Fisher sum becomes

$$\sum_{i,j} \frac{(u_i q_j + p_i v_j)^2}{p_i q_j} = \sum_{i,j} \frac{u_i^2 q_j}{p_i} + 2 \sum_{i,j} u_i v_j + \sum_{i,j} \frac{p_i v_j^2}{q_j} = g_p(u, u) + 0 + g_q(v, v),$$

where the cross term vanishes because $\sum_i u_i = \sum_j v_j = 0$. □

Theorem 8.9 is the compositional content of the statistical family: products of systems are products of metrics, and the classifying invariant of a product is the direct sum of the classifying invariants of the factors. The parallel statement for symplectic products is theorem 9.1 and the decomposition analysis of Section 9.

The uniqueness statement is classical in both senses of the word. The quantum analogue fails.

Theorem 8.10 (Petz-Sudar parametric classification). *Monotone metrics on the space of positive definite density matrices, defined by monotonicity under completely positive trace-preserving maps, are classified by operator monotone functions $f : (0, \infty) \rightarrow (0, \infty)$ with $f(1) = 1$ and the symmetry $f(t) = tf(1/t)$ that makes the bilinear form symmetric. The family of such metrics is parametric and infinite, and no constant multiple of a single metric exhausts it.*

Proof reference. Petz and Sudar [13] classify quantum monotone metrics by operator monotone functions. The metric attached to such a function f is $g_{f,\rho}(A, B) = \text{tr}(A \mathbf{J}_{\rho,f}^{-1}(B))$ where $\mathbf{J}_{\rho,f} = \mathcal{R}_\rho f(\mathcal{L}_\rho \mathcal{R}_\rho^{-1})$ and $\mathcal{L}_\rho$ and $\mathcal{R}_\rho$ are left and right multiplication by the state ρ ; the inverse operator carries the ρ^{-1} scaling that reproduces the classical Fisher metric in the commutative case. The paper uses this classification as the contrast that blocks any transfer of Chentsov uniqueness to the quantum case. □

Theorem 8.10 is a warning about scope. Classical uniqueness in the classification of monotone metrics may not be extended to the quantum family by dependency or analogy: the quantum classification is parametric, not unique. Section 11 returns to the boundary between the two.

9 Decomposition and restricted classification

Complex classical systems are built from simpler ones by products, reductions, and coupling. Irreducibility is relative to the product composition fixed below, and two restricted classification theorems survive for global symplectic objects.

Definition 9.1 (Product composition). The product composition of classical systems is defined componentwise. For symplectic systems, $(M_1 \times M_2, \omega_1 \oplus \omega_2)$ with $\omega_1 \oplus \omega_2 = \text{pr}_1^* \omega_1 + \text{pr}_2^* \omega_2$ and Hamiltonians $H_1 + H_2$. For Poisson systems, the direct sum of bivectors. For statistical systems, the product of simplices with product measures and product stochastic maps. The composition data C of theorem 2.6 is this product.

Proposition 9.2 (Irreducibility of symplectic surfaces). *Every closed symplectic surface of genus $g \geq 0$ is C -irreducible for the product composition.*

Proof. A product decomposition $(M_1 \times M_2, \omega_1 \oplus \omega_2)$ of a 2-dimensional symplectic manifold has $\dim M_1 + \dim M_2 = 2$ with both factors even-dimensional and symplectic, hence of dimension 0 or 2. Positive dimension forces one factor to be the empty product. $\square$

Proposition 9.3 (Symplectic reduction). *Let a compact Lie group G act on a symplectic system (M, ω) by symplectomorphisms with momentum map J . For a regular value μ of J at which the isotropy group G_μ acts freely and properly on $J^{-1}(\mu)$, the reduced space $J^{-1}(\mu)/G_\mu$ carries a unique symplectic form ω_μ such that the pullback along $J^{-1}(\mu) \rightarrow J^{-1}(\mu)/G_\mu$ agrees with $\omega|_{J^{-1}(\mu)}$, and the reduced dynamics of an invariant Hamiltonian is Hamiltonian for ω_μ . The reduction is a decomposition construction on classical systems, and its output is again a classical system of theorem 2.1.*

Proof reference. The construction and its compatibility with dynamics are the symplectic reduction theorem of Marsden and Weinstein [10]. Regularity of μ gives the manifold structure of $J^{-1}(\mu)$ and of the quotient by the isotropy group G_μ , and the uniqueness of ω_μ is the statement that $\ker(\omega|_{J^{-1}(\mu)})$ is spanned by the infinitesimal action directions. $\square$

Reduction is the mechanism by which symmetry data enters the composition layer of a classical system: it removes gauge directions and lowers dimension while preserving the symplectic structure. Existence of decompositions is therefore abundant in the presence of symmetry. Decompositions are generally non-unique.

Example 9.4 (Reduction of the plane). Let $U(1)$ act on $\mathbb{C}^2 \cong \mathbb{R}^4$ with its standard symplectic form by $e^{i\theta} \cdot z = e^{i\theta}z$, with momentum map $J(z) = \frac{1}{2}(|z_1|^2 + |z_2|^2)$. For $\mu > 0$ the level set $J^{-1}(\mu)$ is the three-sphere of radius $\sqrt{2\mu}$ in $\mathbb{C}^2$, and the reduced space $J^{-1}(\mu)/U(1)$ is $\mathbb{CP}^1$ with the Fubini-Study symplectic form normalized to total area $2\pi\mu$; the quotient identifies the Hopf fibers of the level set. The reduced dynamics of any rotation-invariant Hamiltonian is Hamiltonian on that sphere. The example is the elementary case of theorem 9.3: a four-dimensional symplectic system with a one-dimensional symmetry reduces to a two-dimensional one, and the reduction is a decomposition step in the sense of theorem 9.1.

Proposition 9.5 (Reduction of products). *Let G act on a symplectic system (M_1, ω_1) with momentum map J_1 , let (M_2, ω_2) carry the trivial G -action, and let μ be a regular value of J_1 . Then the reduced space of the product $(M_1 \times M_2, \omega_1 \oplus \omega_2)$ at μ is the product of the reduced space of M_1 at μ with M_2 :*

$$(J_1^{-1}(\mu)/G_\mu) \times M_2,$$

with reduced form $\omega_\mu \oplus \omega_2$. Reduction commutes with product composition along the factor that carries no symmetry.

Proof. The momentum map of the product action is $J = J_1 \circ \text{pr}_1$, so $J^{-1}(\mu) = J_1^{-1}(\mu) \times M_2$ and the isotropy group G_μ acts only on the first factor. The quotient is therefore the product of the quotient with M_2 , and the reduced form is the direct sum because the symplectic form of the product splits along the same factors. Uniqueness of the reduced form from theorem 9.3 finishes the argument. $\square$

Theorem 9.5 states the compatibility between the two constructions of this section, product composition and symmetry reduction. What it does not state is any uniqueness for decomposition: the factors of a product can themselves be reducible, and the reduced factor $J_1^{-1}(\mu)/G_\mu$ need not be irreducible even when M_1 is.

Class of classical systems	Classifying invariant	Scope of the classification
Local symplectic manifolds	none (Darboux)	complete at every dimension and trivial
Local Poisson manifolds	rank $2k$ and transverse factor	local normal form at every point
Coadjoint orbits of compact semisimple G	positive Weyl chamber parameter and flag type	homogeneous symplectic manifolds, integral orbits by weights
Toric symplectic manifolds	Delzant polytope	compact connected manifolds with T^n -action
Closed symplectic surfaces	genus and total area	dimension two, complete
Classical statistical models	Fisher metric up to scale	monotone metrics under stochastic maps
Integrable systems near a regular fiber	action angles and period lattice	local, obstructed globally by monodromy

Table 3: The classification inventory of classical systems. Each row is a complete theorem inside its stated scope, and no row claims more than that scope.

Remark 9.6 (Uniqueness of decomposition). No general uniqueness theorem for decomposition of classical systems is claimed. The local decomposition of integrable systems in theorem 7.2 is unique in the sense of action-angle data near a regular fiber, and theorem 7.4 shows that even this restricted uniqueness fails to globalize. Existence and uniqueness of decomposition are realistic goals only in restricted subclasses, and the paper states its restricted theorems with that limitation visible.

The first restricted global classification theorem in the symplectic category concerns toric manifolds.

Definition 9.7 (Toric symplectic manifold). A compact connected toric symplectic manifold is a symplectic manifold (M^{2n}, ω) of dimension $2n$ with an effective Hamiltonian action of the torus T^n whose momentum map $J : M \rightarrow \mathbb{R}^n$ is proper.

Theorem 9.8 (Delzant). *The momentum image $\Delta = J(M) \subset \mathbb{R}^n$ of a compact connected toric symplectic manifold is a convex rational Delzant polytope: convex, rational, simple, and Delzant (the normal vectors of the facets meeting at each vertex form a $\mathbb{Z}$ -basis of $\mathbb{Z}^n$). Two compact connected toric symplectic manifolds are equivariantly symplectomorphic up to an automorphism of the acting torus T^n if and only if their Delzant polytopes are related by an element of $GL(n; \mathbb{Z}) \ltimes \mathbb{R}^n$. When the torus action is fixed pointwise the equivalence narrows to translations of the polytope in $\mathbb{R}^n$ alone. Conversely, every Delzant polytope arises from such a manifold.*

Proof reference. Delzant [4]. The polytope is the complete invariant: equivariant symplectic classification of toric manifolds is classification of Delzant polytopes. It is the model of a restricted classification theorem with a finite-dimensional combinatorial invariant, the exact analogue on the global symplectic side of the Dynkin list of theorem 6.2. $\square$

Example 9.9 (Toric surfaces). In dimension two the Delzant polygons are the polygons with rational edge slopes meeting in lattice bases. The moment polygon of $(S^2, r\omega_{\text{area}})$ is an interval of length r , and the moment polygon of the one-point blowup of $(\mathbb{CP}^2, \omega)$ is a triangle with one corner cut. Two toric surfaces are equivariantly symplectomorphic if and only if their polygons coincide up to lattice equivalence, so the classification is a combinatorial list of polygons.

Table 3 is the classification inventory the evidence supports. The rows are disjoint regimes, the invariants are finite or parametric data in every case, and the empty row is the global classification of closed symplectic manifolds beyond surfaces and the toric subclass.

10 Computational investigation

Three computational experiments and the executable taxonomy models accompanying the paper illustrate the normal forms of Sections 3 to 8 and supply checked examples of the invariant computations. Computational evidence is not proof, and the theorems stand on their proofs and citations rather than on these runs.

The Darboux chart experiment takes a perturbed two-form $\omega = \omega_0 + d\alpha$ on a ball in $\mathbb{R}^4$ and integrates the Moser homotopy of theorem 3.2, producing a diffeomorphism that pulls ω back to ω_0 . The residual $\|\varphi^*\omega - \omega_0\|$ is computed and reported. On the tested instance (quadratic perturbation with scale 0.05, twelve sample points, four hundred Runge-Kutta steps, complex-step Jacobians at step 10^{-20}) the maximum entrywise residual $\|J^T\omega(\varphi(x_0))J - \omega_0(x_0)\|_{\max}$ is 2.4×10^{-15} , at double precision floor. The experiment evidences the constructive content of the proof of theorem 3.3: the homotopy exists as an ordinary differential equation and its numerical integration is stable on the tested examples.

The rank stratification experiment applies the Weinstein splitting of theorem 4.1 to the Lie-Poisson structure of $\mathfrak{so}(3)^*$. The program computes the rank of the Poisson tensor at sample points, extracts the symplectic leaf dimension at each point, and verifies the rank stratification of theorem 4.3: rank two on the spheres of radii 0.5, 1.0, and 2.7 with a two-dimensional symplectic factor and a one-dimensional transverse factor, and rank zero at the origin with a three-dimensional transverse factor. The experiment evidences the rank stratification claim in one explicit model.

The Fisher monotonicity experiment checks the monotonicity of the Fisher metric under stochastic maps, the hypothesis of theorem 8.4. The program computes g_p and the pullback T^*g_{Tp} for random column-stochastic matrices T and verifies $T^*g_{Tp} \leq g_p$ by comparing the quadratic forms. On two hundred sampled trials with five outcomes, the minimum eigenvalue of $g_p - T^*g_{Tp}$ is -5.6×10^{-15} , the round-off signature of a positive semidefinite difference, and the minimum slack $g_p(u, u) - g_{Tp}(Tu, Tu)$ over ten thousand random tangent vectors is 5.4×10^{-2} . The tangent vectors were drawn as Gaussian vectors projected to the tangent space ($\sum_i u_i = 0$) without normalization, so the strictly positive slack reflects that the samples miss the near-null directions of the difference matrix. The experiment evidences the monotonicity direction of Chentsov’s theorem on finite samples. It does not evidence the uniqueness direction, and no claim of uniqueness rests on the computation.

The Haskell models in `src/vol-05-classical-systems/` declare instances of the shared taxonomy classes for symplectic systems, Poisson systems, Hamiltonian systems, and classical statistical systems, with names mirroring the definitions above and the companion verification targets. They compute witness data for the normal forms: standard symplectic matrices, rank splits of Poisson tensors, the Kirillov-Kostant-Souriau form on a two-dimensional orbit, and the Fisher metric on probability vectors. The models are executable, not machine-checked proofs. Appendix A maps each computational claim to its file and function.

The core of the first experiment is short enough to reproduce. The perturbation scale is `EPS = 0.05`, `alpha` is the one-form with quadratic coefficients whose exterior derivative perturbs ω_0 , and `flow_time1` integrates the Moser field with fourth-order Runge-Kutta from $t = 0$ to $t = 1$. The Moser field is the pointwise solution of a linear system: with $\iota_{X_t}\omega_t = -\alpha_t$ and the matrix convention $M_{ij} = \omega_t(e_i, e_j)$, the contraction reads $MX = \alpha_t$, so the routine solves $MX = a$ with $a = \text{EPS} \cdot \alpha(x)$.

```

def moser_field(x, t):
    # solve omega_t(x) X = alpha(x), the matrix form of
    # iota_X omega_t = -alpha with M skew-symmetric
    M = omega_t(x, t)
    a = EPS * alpha(x)
    return np.linalg.solve(M, a)

def test_chart_pullback(n_samples=12, h=1e-20):
    worst = 0.0
    for _ in range(n_samples):
        x0 = rng.normal(scale=0.3, size=DIM)
        x1 = flow_time1(x0)
        J = np.zeros((DIM, DIM))
        for k in range(DIM):
            e = np.zeros(DIM, dtype=complex)
            e[k] = 1j * h
            J[:, k] = np.imag(flow_time1(x0.astype(complex) + e)) / h
        resid = J.T @ omega(x1) @ J - omega_0(x0)
        worst = max(worst, np.max(np.abs(resid)))
    return worst

```

Listing 1: Core of the Darboux chart experiment (darboux_chart.py).

Listing 1 is the complete pullback test: the flow is integrated with fourth-order Runge-Kutta, the Jacobian of the time-one map is taken by complex-step differentiation so that the residual is not dominated by difference truncation, and the pullback form is compared to ω_0 at the base point x_0 . The step $h = 10^{-20}$ is representable in IEEE double precision, whose smallest normal number is near 10^{-308} , so the complex-step quotient does not underflow on the tested platform. The other two experiments follow the same pattern, a direct invariant computation checked against the predicted inequality or rank.

11 Limitations and open questions

The classification results above hold exactly in the regimes of the established mathematics they rest on.

Global classification of closed symplectic manifolds is open. The invariants of Section 5 classify in restricted regimes: surfaces by area and genus, toric manifolds by Delzant polytopes, homogeneous manifolds by weight data. Beyond these regimes the diffeomorphism classification of the underlying manifold is itself incomplete in dimension four and above, and the symplectic classification contains it. No claim of the paper addresses that gap.

The classification of Poisson structures beyond local models is open. The Weinstein splitting theorem classifies the local type at each point, and the leaf foliation organizes the global object, but the topology of leaf spaces, the structure of singular leaves, and the classification of Poisson manifolds with prescribed leaf space are not addressed by the local theory. This paper makes no claim about them.

Decomposition uniqueness is open in general. Theorem 7.4 exhibits the obstruction inside one of the best behaved subclasses: action-angle monodromy prevents global decomposition of integrable systems whose local decomposition is complete. Characterizing the subclasses with unique decomposition is a genuine gap in the classical theory.

The bridge between classical and quantum families is conjectural. Deformation quantization associates to a Poisson manifold a family of noncommutative algebras, and it is one of the candidate

constructions connecting the symplectic family to the operator-algebra family. No equivalence theorem connects the two families, and the paper treats the connection as a programmatic conjecture: every statement crossing the classical and quantum boundary is conditional on that conjecture and is marked as such. The Petz-Sudar classification of Theorem 8.10 shows concretely that uniqueness phenomena do not transfer: the classical uniqueness of the Fisher metric has a parametric quantum shadow.

The equivalence relation of the series is not fixed here. This paper classifies under mathematical equivalence of theorem 2.5, which preserves the symplectic form, the Poisson bracket, or the statistical structure exactly. Physical equivalence for classical systems must be coarser, and identifying two classical systems as physically equivalent requires naming the preserved layers and the observational content. The paper does neither, and the classification statements above are statements about mathematical objects.

The computational experiments of Section 10 demonstrate that the normal forms are computable on examples; no claim of the paper depends on them.

The open questions the classical sector poses to the series are the following.

1. Which global invariants beyond $[\omega]$, the characteristic classes, and the capacities complete the classification of closed symplectic manifolds in dimension four? The inventory of Section 5 is partial in that regime and complete nowhere else.
2. Which subclasses of classical systems admit unique decomposition into C -irreducible factors? The integrable subclass admits local decomposition and fails global decomposition by monodromy, and the boundary between the two behaviors is not characterized.
3. What is the classification of Poisson structures with prescribed leaf space? The Weinstein splitting theorem classifies the local type and leaves the global organization of singular leaves open.
4. Does deformation quantization connect the symplectic family to the operator-algebra family in any sense stronger than a programmatic correspondence? Every statement of this paper that crosses that boundary is conditional on the conjecture.
5. Which coarsenings of mathematical equivalence are the physically correct equivalences for classical systems? The answer must name the preserved layers and the observational content, and the paper fixes neither.

12 Conclusion

Classical physical systems admit classification exactly where the invariants are finite, parametric, or combinatorial, and the paper has exhibited each type of success in this domain. Local symplectic classification is complete and trivial (Darboux), and the classification content is global. Local Poisson classification is a rank stratification with a transverse factor (Weinstein). Homogeneous symplectic classification is the Dynkin and highest-weight theory of Lie algebras (coadjoint orbits). Global classification succeeds for toric manifolds (Delzant polytopes) and for surfaces (genus and area). The classical statistical family carries a unique monotone metric up to scale (Chentsov), and the quantum analogue is parametric rather than unique (Petz-Sudar).

The negative results are theorems of the same standing. There are no local symplectic invariants. Global action-angle coordinates fail by monodromy. Classical uniqueness phenomena do not extend to

the quantum family. These are content, not caveats: they delimit the regimes in which classification theorems can be expected, and they block the false transfers that analogy would suggest.

The classification inventory of Table 3 is complete for the regimes it names and silent elsewhere, and the next classical results will come either from widening a regime (which subclasses of integrable systems decompose uniquely) or from deepening an obstruction (what exactly monodromy forbids beyond global action-angle coordinates). The evidence supports restricted classification theorems plus explicit obstructions in the classical domain. It does not support a universal periodic table of classical systems, and the paper does not claim one.

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A Code evidence

Table 4 maps each computational claim of Section 10 to the file and function that demonstrates or tests it. Every entry points at code in `src/vol-05-classical-systems/`. The claims are existence claims about computations, and the code demonstrates exactly the claim stated.

Computational claim	File	Function or test
Moser homotopy produces a Darboux chart	<code>darboux_chart.py</code>	<code>flow_time1</code> and <code>test_chart_pullback</code>
Poisson rank stratification on $\mathfrak{so}(3)^*$	<code>weinstein_splitting.py</code>	<code>split_bivector</code> and <code>test_rank_stratification</code>
Fisher metric is monotone under stochastic maps	<code>fisher_monotonicity.py</code>	<code>pullback_comparison</code> and <code>test_monotonicity</code>
Standard symplectic data and witness functions	<code>Instances.hs</code>	<code>darboux</code> , <code>weinsteinSplitting</code> , and <code>rankChecks</code>
KKS form computation on a two-dimensional orbit	<code>Instances.hs</code>	<code>kksSymplecticForm</code>
Fisher metric on probability vectors	<code>Instances.hs</code>	<code>fisherMetric</code>

Table 4: Code evidence for the computational claims of Section 10. Each row names one claim and the code that demonstrates it.